PROBLEM OF THE WEEK

Solution of Problem No. 11 (Spring 2006 Series)

Problem: Three lines in space, not coplanar, intersect in a common point O. Given a point P not on any of those lines, characterize the plane through P that cuts off a tetrahedron with vertex O of minimal volume.

Solution (by Steven Landy, IUPUI Physics; edited by the Panel)

Let $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ be unit vectors from O along the lines. They form a basis for $\mathbb{R}^3$. Let $a\vec{e}_1$, $a\vec{e}_2$, and $a\vec{e}_3$ be the points where a plane is hit by the lines. Let $\vec{p} = (p_1, p_2, p_3)$ be a vector from O to P. If P is in the plane, we have

$$p_1 = \alpha_1 a_1 \qquad p_2 = \alpha_2 a_2 \qquad p_3 = \alpha_3 a_3$$

with

$$\alpha_1 + \alpha_2 + \alpha_3 = 1.$$

The volume of the tetrahedron is

$$V = \frac{a_1 a_2 a_3}{6} \left|\vec{e}_1 \cdot (\vec{e}_2 \times \vec{e}_3)\right|.$$

So, we want to minimize

$$a_1 a_2 a_3 = \frac{p_1 p_2 p_3}{\alpha_1 \alpha_2 \alpha_3}$$

with constraint $\alpha_1 + \alpha_2 + \alpha_3 = 1$. In other words, we want to maximize $\alpha_1\alpha_2\alpha_3$ subject to the same constraint. This occurs when $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$. Thus P must be the centroid of triangle with intercepts $a_1\vec{e}_1$ $a_2\vec{e}_2$ $a_3\vec{e}_3$ or said the other way, the intercepts must be $3p_1\vec{e}_1$, $3p_2\vec{e}_2$, $3p_3\vec{e}_3$.

Also solved by:

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