PROBLEM OF THE WEEK Solution of Problem No. 12 (Spring 2006 Series)

Problem: Evaluate

$$S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{mn^2}{2^n(n2^m + m2^n)}.$$

Solution (by Georges Ghosn, Quebec; edited by the Panel)

The double series converges. Indeed $\frac{mn^2}{2^n(n2^m+m2^n)} < \frac{mn^2}{2^n \cdot n2^m} = \frac{mn}{2^n \cdot 2^m}$. Next, $\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2}f'\left(\frac{1}{2}\right), \text{ where } f(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \text{ which converges on } (-1,1); \text{ and}$ therefore $xf'(x) = \sum_{k=1}^{+\infty} kx^k = \frac{x}{(1-x)^2}.$ Hence

$$\lim_{M \to \infty} \sum_{m=1}^{M} \left(\frac{m}{2^m} \cdot \lim_{N \to \infty} \sum_{n=1}^{N} \frac{n}{2^n} \right) = \lim_{M \to \infty} \sum_{m=1}^{M} \left(\frac{m}{2^m} \cdot \frac{1}{2} f'\left(\frac{1}{2}\right) \right) = \left(\frac{1}{2} f'\left(\frac{1}{2}\right) \right)^2 = 4.$$

Therefore, from the comparison test we deduce that

$$S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{mn^2}{2^n(n2^m + m2^n)} \quad \text{converges} \quad \text{and} \quad S \le 4$$

We pose $a_n = \frac{2^n}{n}$. Then $\frac{mn^2}{2^n(n2^m + m2^n)} = \frac{1}{a_n(a_m + a_n)}$. Since the double series converges and has only positive terms we can swap the summations. Therefore,

$$S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{a_n(a_m + a_n)} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{a_n(a_m + a_n)}$$
$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{a_m(a_n + a_m)}.$$
 (renaming *m* and *n*)

Finally

$$S = \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{1}{a_n(a_m + a_n)} + \frac{1}{a_m(a_n + a_m)} \right)$$
$$= \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{a_n a_m} = \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m \cdot n}{2^m \cdot 2^n} = \frac{1}{2} \cdot 4 = 2.$$

Also, at least partially solved by:

Undergraduates: Alan Bernstein (Jr. ECE), Ramul Kumar (Fr. E)

<u>Graduates</u>: Tom Engelsman (ECE)

<u>Others</u>: Prithwijit De (Ireland), Hoan Duong (San Antonio College), Duc Van Huynh (Armstrong Atlantic State U.), Steven Landy (IUPUI Physics staff), Sandeep Sarat (Johns Hopkins U.)