## PROBLEM OF THE WEEK

Solution of Problem No. 12 (Spring 2006 Series)

Problem: Evaluate

$$
S=\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m n^{2}}{2^{n}\left(n 2^{m}+m 2^{n}\right)}
$$

Solution (by Georges Ghosn, Quebec; edited by the Panel)
The double series converges. Indeed $\frac{m n^{2}}{2^{n}\left(n 2^{m}+m 2^{n}\right)}<\frac{m n^{2}}{2^{n} \cdot n 2^{m}}=\frac{m n}{2^{n} \cdot 2^{m}}$. Next, $\sum_{n=1}^{\infty} \frac{n}{2^{n}}=\frac{1}{2} f^{\prime}\left(\frac{1}{2}\right)$, where $f(x)=\sum_{k=0}^{\infty} x^{k}=\frac{1}{1-x}$ which converges on $(-1,1)$; and therefore $x f^{\prime}(x)=\sum_{k=1}^{+\infty} k x^{k}=\frac{x}{(1-x)^{2}}$.
Hence

$$
\lim _{M \rightarrow \infty} \sum_{m=1}^{M}\left(\frac{m}{2^{m}} \cdot \lim _{N \rightarrow \infty} \sum_{n=1}^{N} \frac{n}{2^{n}}\right)=\lim _{M \rightarrow \infty} \sum_{m=1}^{M}\left(\frac{m}{2^{m}} \cdot \frac{1}{2} f^{\prime}\left(\frac{1}{2}\right)\right)=\left(\frac{1}{2} f^{\prime}\left(\frac{1}{2}\right)\right)^{2}=4
$$

Therefore, from the comparison test we deduce that

$$
S=\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m n^{2}}{2^{n}\left(n 2^{m}+m 2^{n}\right)} \quad \text { converges } \quad \text { and } \quad S \leq 4
$$

We pose $a_{n}=\frac{2^{n}}{n}$. Then $\frac{m n^{2}}{2^{n}\left(n 2^{m}+m 2^{n}\right)}=\frac{1}{a_{n}\left(a_{m}+a_{n}\right)}$. Since the double series converges and has only positive terms we can swap the summations. Therefore,

$$
\begin{aligned}
S & =\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{a_{n}\left(a_{m}+a_{n}\right)}=\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{a_{n}\left(a_{m}+a_{n}\right)} \\
& =\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{a_{m}\left(a_{n}+a_{m}\right)} . \quad(\text { renaming } m \text { and } n)
\end{aligned}
$$

Finally

$$
\begin{aligned}
S & =\frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty}\left(\frac{1}{a_{n}\left(a_{m}+a_{n}\right)}+\frac{1}{a_{m}\left(a_{n}+a_{m}\right)}\right) \\
& =\frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{a_{n} a_{m}}=\frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m \cdot n}{2^{m} \cdot 2^{n}}=\frac{1}{2} \cdot 4=2 .
\end{aligned}
$$

Also, at least partially solved by:

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