PROBLEM OF THE WEEK

Solution of Problem No. 14 (Spring 2006 Series)

Problem: Let $P(x)$ be a polynomial of degree $n \geq 2$ with real coefficients of the form

$$P(x) = ax^n + bx^{n-1} + cx^{n-2} + \cdots, \quad a \neq 0.$$

Show that if $b^2 - \frac{2n}{n-1}ac < 0$, then $P(x)$ can not have more than $n-2$ real zeros.

Solution (by Prithwijit De, Ireland; edited by the Panel)

Suppose $P(x)$ has more than $(n-2)$ real roots. Since the number of complex roots of a polynomial with real coefficients is even, $P(x)$ must have n real roots. Let the roots be $x_1, \cdots, x_n$. Then by the Cauchy–Schwarz inequality, we have the following:

$$\left(\sum_{i=1}^{n} x_i\right)^2 \leq n\left(\sum_{i=1}^{n} x_i^2\right).$$

Also, $\left(\sum_{i=1}^{n} x_i\right)^2 = \frac{b^2}{a^2}$ and $\sum_{i=1}^{n} x_i^2 = \frac{b^2 - 2ac}{a^2}$. Substituting these expressions in the inequality yields

$$n(b^2 - 2ac) - b^2 \geq 0 \Rightarrow b^2 - \frac{2n}{n-1}ac \geq 0$$

Therefore, if the hypothesis of the problem holds then the number of real roots of $P(x)$ will not be more than $n-2$.

Also, at least partially solved by:

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