PROBLEM OF THE WEEK

Solution of Problem No. 2 (Spring 2006 Series)

Problem: Find all positive real numbers a such that

$$\sqrt[3]{3+\sqrt{a}}+\sqrt[3]{3-\sqrt{a}}$$

is a positive integer.

Solution (by Bhilahari Jeevanesan, Germany; edited by the Panel)

In the identity $x^3+y^3=\{(x+y)^2-3xy\}(x+y)$ set $x=\sqrt[3]{3+\sqrt{a}}$, $y=\sqrt[3]{3-\sqrt{a}}$ and $x+y=n$, which is expected to be an integer. The resulting equation then is

$$6=\left\{n^2-3\sqrt[3]{9-a}\right\}n.$$

Hence

$$a=9-(n^2/3-2/n)^3,$$

from where it follows that for $n=1$, $a=\dfrac{368}{27}$ and for $n=2$, $a=\dfrac{242}{27}$. Since $(n^2/3-2/n)^3>9$ when $n\geq 3$, these are also the only possible solutions for positive n and a.

Now, one can see directly that those two values of a are indeed solutions.

At least partially solved by:

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