PROBLEM OF THE WEEK

Solution of Problem No. 3 (Spring 2006 Series)

Problem: Let $a_1 = \sqrt{2}, \quad a_2 = \left(\sqrt{2}\right)^{a_1}$, and

$$a_n = \left(\sqrt{2}\right)^{a_{n-1}}, \quad n = 2, 3, \ldots .$$

Show that $\lim_{n\to\infty} a_n$ exists and determine its value.

Solution (by Tomek Czajka, CS graduate student; edited by the Panel)

Let $a_0 = 1$. Then $a_1 = \sqrt{2}^{a_0}$. If $a_n < 2$, then $a_{n+1} = \sqrt{2}^{a_n} < \sqrt{2}^2 = 2$. Since $a_0 < 2$, by induction, $a_n < 2$ for all n. If $a_n < a_{n+1}$, then $a_{n+1} = \sqrt{2}^{a_n} < \sqrt{2}^{a_{n+1}} = a_{n+2}$. Since $a_0 = 1 < \sqrt{2} = a_1$, by induction we get that the sequence (a_n) increases. Since (a_n) increases, starts from 1 and is bounded by 2, it has a limit between 1 and 2 (inclusive). Let $a = \lim_{n\to\infty} a_n$. We have $1 \le a \le 2$. Also:

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2}^{a_n} = \sqrt{2}^{\lim_{n\to\infty} a_n} = \sqrt{2}^a$$

$$a^2 = 2^a$$

$$2 \ln a = a \ln 2$$

$$2 \ln a - a \ln 2 = 0$$

Let $f(x) = 2 \ln x - x \ln 2$ for $x > 0$. Then $f(a) = 0$ and $f(2) = 0$. For $1 \le x \le 2$, we have $f'(x) = 2/x - \ln 2 > 2/2 - \ln e = 0$. Therefore f increases on the interval [1,2] and thus is injective on this interval. Since $a \in [1, 2]$ and $f(a) = f(2)$, we must have $a = 2$.

Answer:

$$\lim_{n\to\infty} a_n = 2.$$

At least partially solved by:

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