PROBLEM OF THE WEEK

Solution of Problem No. 5 (Spring 2006 Series)

Problem: Without using a computational device, prove that

$$\left(\tan\frac{3\pi}{11} + 4\,\sin\frac{2\pi}{11}\right)^2$$

is an integer.

Solution (by Tom Czajka, CS graduate student; edited by the Panel)

Let $z = e^{\pi i/11}$. Then:

$z^{11} = e^{\pi i} = -1$

$(1+z)(1 - z + z^2 - z^3 + z^4 - z^5 + z^6 - z^7 + z^8 - z^9 + z^{10} = 1 + z^{11} = 0$

$1 - z + z^2 - z^3 + z^4 - z^5 + z^6 - z^7 + z^8 - z^9 + z^{10} = 0$

$(1 - z^5)(z - z^2 + z^3 - z^4 + z^5) = 1$

By Euler's formula,

$$z^k = \cos\frac{k\pi}{11} + i\,\sin\frac{k\pi}{11}.$$

Therefore,

$$\begin{aligned}
\sin\frac{k\pi}{11} &= \mathrm{Im}(z^k) = \frac{z^k - \overline{z^k}}{2i} = \frac{i}{2}(z^{-k} - z^k) \\
\cos\frac{k\pi}{11} &= \mathrm{Re}\,(z^k) = \frac{z^k + \overline{z^k}}{2} = \frac{1}{2}(z^{-k} + z^k) \\
\tan\frac{k\pi}{11} &= \frac{\sin(k\pi/11)}{\cos(k\pi/11)} = i\frac{z^{-k} - z^k}{z^{-k} + z^k} = i\frac{1 - z^{2k}}{1 + z^{2k}} \\
\tan\frac{3\pi}{11} &= i\frac{1 - z^6}{1 + z^6} = i\frac{z^5 - z^{11}}{z^5 + z^{11}} = -i\frac{1 + z^5}{1 - z^5} = -i\left(\frac{2}{1 - z^5} - 1\right) \\
&= -i\left(2(z - z^2 + z^3 - z^4 + z^5) - 1\right) = i(1 - 2z + 2z^2 - 2z^3 + 2z^4 - 2z^5) \\
4\sin\frac{2\pi}{11} &= 2i(z^{-2} - z^2) = -i(2z^2 + 2z^9) \\
\tan\frac{3\pi}{11} + 4\sin\frac{2\pi}{11} &= i(1 - 2z - 2z^3 + 2z^4 - 2z^5 - 2z^9).
\end{aligned}$$

Therefore, using the equalities in the beginning of the proof, we get

$$\left(\tan\tfrac{3\pi}{11} + 4\sin\tfrac{2\pi}{11}\right)^2 = -4(1/2 - z - z^3 + z^4 - z^5 - z^9)^2$$

$= -4((1/2)^2 + z^2 + z^6 + z^8 + z^{10} + z^{18} - z - z^3 + z^4 - z^5 - z^9$

$$
\begin{aligned}
&\qquad +2zz^3 - 2zz^4 + 2zz^5 + 2zz^9 - 2z^3z^4 + 2z^3z^5 + 2z^3z^9 - 2z^4z^5 - 2z^4z^9 + 2z^5z^9) \\
&= -4(1/4) + z^2 + z^6 + z^8 + z^{10} + z^{18} - z - z^3 + z^4 - z^5 - z^9 \\
&\qquad +2z^4 - 2z^5 + 2z^6 + 2z^{10} - 2z^7 + 2z^8 + 2z^{12} - 2z^9 - 2z^{13} + 2z^{14}) \\
&= -4(1/4 - z + z^2 - z^3 + 3z^4 - 3z^5 + 3z^6 - 2z^7 + 3z^8 - 3z^9 + 3z^{10} + 2z^{12} - 2z^{13} + 2z^{14} + z^{18}) \\
&= -4(1/4 - z + z^2 - z^3 + 3z^4 - 3z^5 + 3z^6 - 2z^7 + 3z^8 - 3z^9 + 3z^{10} - 2z + 2z^2 - 2z^3 - z^7) \\
&= -4(1/4 - 3 + 3(1 - z + z^2 - z^3 + z^4 - z^5 + z^6 - z^7 + z^8 - z^9 + z^{10})) = -4(1/4 - 3) = 11
\end{aligned}
$$

Also, at least partially solved by:

Graduates: Tom Engelsman (ECE)

Others: Georges Ghosn (Quebec), Kevin Laster (Indiana), Babenko Michail (Stavropol, Russia), Sandeep Sarat (John Hopkins U.)