PROBLEM OF THE WEEK

Solution of Problem No. 6 (Spring 2006 Series)

Problem: Let C be the curve whose equation is

$$y = P(x),$$

where $P(x)$ is a real polynomial having at least one real zero $x_0 \neq 0$.

Prove that there exists a point on the curve C, different from $(x_0, 0)$, whose distance to $(0,0)$ is $|x_0|$.

Solution (by the Panel)

Intuitively, the statement is clear. Let K be the circle with center $(0,0)$ and radius $|x_0|$. Then K and C have at least one common point $(x_0, 0)$ and they are not tangent to each other at this point (because K has a vertical tangent, C has a finite slope $P'(x_0)$). Therefore, C enters K, and it has to leave somewhere because $P(x)$ is defined everywhere.

To make those arguments precise, introduce the function

$$f(x) = P^2(x) + x^2 - {x_0}^2.$$

All common points of C and K solve $f(x) = 0$ and vice–versa. Now,

$$f(x_0) = 0, \qquad \lim_{x \to \pm\infty} f(x) = \infty,$$
$$f'(x_0) = 2x_0 \neq 0.$$

Therefore, $f(x)$ must take a negative value somewhere, because otherwise x_0 would be a global minimum, and that would contradict $f'(x_0) \neq 0$. If that happens for $x_1 > x_0$, then we apply the intermediate value theorem for $f(x)$ on $[x_1, A]$, where $A > x_1$ is such that $f(A) > 0$ (such an A exists because $f(x_1) \to \infty$, as $x \to \infty$). If $x_1 < x_0$, then we do the same thing on $[B, x_1]$, where $B_1 < x_1$ and $f(B) > 0$. In either case, we get another zero of $f(x)$.

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