PROBLEM OF THE WEEK

Solution of Problem No. 9 (Spring 2006 Series)

Problem: Let $A > 0$ be a rational number. Show that A is the area of a right triangle with rational sides if and only if there exist three rational numbers u, v and w, so that

$$u^2 - v^2 = v^2 - w^2 = A.$$

Solution (by Nguyen Nguyen T.K., San Antonio College; edited by the Panel)

Let A be the area of the right triangle T with rational sides a, b, c ($a \le b < c$). Then $a^2 + b^2 = c^2$ and $A = \frac{ab}{2}$. Let $u = \frac{a+b}{2}$, $v = \frac{c}{2}$, and $w = \frac{b-a}{2}$. Then u, v, and w are rational numbers, and we have

$$A = \frac{ab}{2} = \frac{a^2 + 2ab + b^2}{4} - \frac{a^2 + b^2}{4} = \left(\frac{a+b}{2}\right)^2 - \left(\frac{c}{2}\right)^2 = u^2 - v^2,$$

and

$$A = \frac{ab}{2} = \frac{a^2 + b^2}{4} - \frac{a^2 - 2ab + b^2}{4} = \left(\frac{c}{2}\right)^2 - \left(\frac{b-a}{2}\right)^2 = v^2 - w^2.$$

On the other hand, let $u > v > w \ge 0$ be rational numbers such that

$$u^2 - v^2 = v^2 - w^2 = A, \qquad (*)$$

for some (rational number) A. Let $a = u - w, b = u + w, c = 2v$. Then a, b, c are rational numbers and since $u^2 + w^2 = 2v^2$ by (*), we have:

$$a^2 + b^2 = (u - w)^2 + (u + w)^2 = 2(u^2 + w^2) = 4v^2 = c^2.$$

Since $u^2 - w^2 = 2(v^2 - w^2) = 2A$ by (*), we have:

$$\frac{ab}{2} = \frac{(u - w) \cdot (u + w)}{2} = \frac{u^2 - w^2}{2} = A.$$

Therefore, a, b, c are rational sides of a right triangle of area A.

At least partially solved by:

Undergraduates: Alan Bernstein (Jr. ECE)

Graduates: Tom Engelsman (ECE)

Others: Rebecca & David Bobick (W. Lafayette), Prithwijit De (Ireland), Georges Ghosn (Quebec), Bhilahari Jeevanesan (Germany), Tim Lee (Junior, RPI), Babenko Michail (Stavropol, Russia), Sandeep Sarat (Johns Hopkins U.)