PROBLEM OF THE WEEK
Solution of Problem No. 1 (Spring 2007 Series)

Problem:

Show that

$$\sum_{k=1}^{n} \frac{a_k b_k}{1+a_k+b_k} \leq \frac{\sum\limits_{k=1}^{n} a_k \sum\limits_{k=1}^{n} b_k}{1+\sum\limits_{k=1}^{n} a_k + \sum\limits_{k=1}^{n} b_k}$$

for any positive a_k, b_k $(k = 1, \ldots n)$.

Solution (by Georges Ghosn, Quebec)

For $n = 1$ the result is clear. Let's prove the inequality for $n = 2$, which is $\frac{ab}{1+a+b} + \frac{cd}{1+c+d} \leq \frac{(a+c)(b+d)}{1+a+c+b+d}$ for any positive a, b, c, d. Indeed

$$\begin{aligned}
&(a+c)(b+d) - \frac{(1+a+c+b+d)}{1+a+b}ab - \frac{(1+a+c+b+d)}{1+c+d}cd \\
&= ab+ad+bc+cd - \left(1+\frac{c+d}{1+a+b}\right)ab - \left(1+\frac{a+b}{1+c+d}\right)cd \\
&= ad+bc - \frac{(c+d)ab}{1+a+b} - \frac{(a+b)cd}{1+c+d} \\
&= \frac{ad(1+a+d)+bc(1+b+c)+(ad-bc)^2}{(1+a+b)(1+c+d)} > 0
\end{aligned}$$

Now by induction suppose the inequality holds for n. Then

$$\sum_{k=1}^{n+1} \frac{a_k b_k}{1+a_k+b_k} = \sum_{k=1}^{n} \frac{a_k b_k}{1+a_k+b_k} + \frac{a_{n+1}b_{n+1}}{1+a_{n+1}b_{n+1}}$$

$$\leq \frac{\left(\sum\limits_{k=1}^{n} a_k\right)\left(\sum\limits_{k=1}^{n} b_k\right)}{1+\left(\sum\limits_{k=1}^{n} a_k\right)+\left(\sum\limits_{k=1}^{n} b_k\right)} + \frac{a_{n+1}b_{n+1}}{1+a_{n+1}+b_{n+1}} \quad \text{(By the induction hypothesis)}$$

$$\leq \frac{\left(\sum\limits_{k=1}^{n+1} a_k\right)\left(\sum\limits_{k=1}^{n+1} b_k\right)}{1+\left(\sum\limits_{k=1}^{n+1} a_k\right)+\left(\sum\limits_{k=1}^{n+1} b_k\right)}. \quad \text{Because of the case } n=2 \text{ with}$$

$$a = \sum_{k=1}^{n} a_k,\ \sum_{k=1}^{n} b_k,\ c = a_{n+1} \text{ and } d = b_{n+1}.$$

Therefore, the inequality holds for any n.

Also solved by:

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