PROBLEM OF THE WEEK
Solution of Problem No. 10 (Spring 2007 Series)

Problem: Determine all positive integers n such that exactly $n/3$ positive integers are $< n$ and relatively prime to n.

Solution (by Kishin B. Sadarangani, ULPGC, Spain)

By using Euler's totient function, the number of positive integers $< n$ that are relatively prime to n, where 1 is counted as being relatively prime to all numbers is given by $\Phi(n)$. Moreover, if

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$$

is the factorization of n, then

$$\Phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right)$$

In our case, $\Phi(n) = \dfrac{n}{3}$ and, consequently,

$$n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right) = \frac{n}{3}.$$

Thus,

$$3(p_1 - 1)(p_2 - 1)\dots(p_k - 1) = p_1 p_2 \dots p_k. \qquad (1)$$

By the uniqueness of the factorization of an integer, we have that some p_1 is equal to 3. Without loss of generality we suppose that $p_1 = 3$.
From (1) we get

$$2(p_2 - 1)(p_3 - 1)\dots(p_k - 1) = p_2 p_3 \dots p_k.$$

By using the same reasoning, we can suppose that $p_2 = 2$, and this give us

$$(p_3 - 1)(p_4 - 1)\dots(p_k - 1) = p_3 p_4 \dots p_k.$$

The last expression says us that the factorization of n only contains to the primes 2 and 3. Consequently, the integers of the form $n = 2^{\alpha_1} 3^{\alpha_2}$ with $\alpha_1, \alpha_2 \in N$ solve our problem.

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