PROBLEM OF THE WEEK

Solution of Problem No. 11 (Spring 2007 Series)

Problem: Given triangle with vertices A, B, C. Points A_1, B_1, C_1 are chosen on sides $\overline{BC}, \overline{CA}, \overline{AB}$ resp., such that the centroid of $\triangle A_1B_1C_1$ coincides with the centroid of $\triangle ABC$ and that the ratio area($\triangle A_1B_1C_1$)/area($\triangle ABC$) is minimal. Determine with proof, the location of A_1, B_1, C_1.

Solution (by Pete Kornya, Ivy Tech faculty, Bloomington, IN)

Let $\underline{a}, \underline{b}, \underline{c}, \underline{a}_1, \underline{b}_1, \underline{c}_1$ be the position vectors of, respectively, A, B, C, A_1, B_1, C_1 which are assumed to be points in R^3. Let $\underline{c} = \underline{0}$. There are $0 \le \lambda_1, \lambda_2, \lambda_3 \le 1$ such that $\underline{a}_1 = \lambda_1 \underline{b}$, $\underline{b}_1 = (1-\lambda_2)\underline{a}$ and $\underline{c}_1 = \lambda_3 \underline{a} + (1-\lambda_3)\underline{b}$. Equating the position vectors of the centroids of $\triangle ABC$ and $\triangle A_1B_1C_1$

$$\frac{1}{3}(\underline{a}+\underline{b}) = \frac{1}{3}\Big[\lambda_1\underline{b} + (1-\lambda_2)\underline{a} + \lambda_3\underline{a} + (1-\lambda_3)\underline{b}\Big]$$

produces $(\lambda_3 - \lambda_2)\underline{a} + (\lambda_1 - \lambda_3)\underline{b} = \underline{0}$. Since $\underline{a}, \underline{b}$ are linearly independent, $\lambda_3 - \lambda_2 = \lambda_1 - \lambda_3 = 0$. Let $\lambda = \lambda_1 = \lambda_2 = \lambda_3$. Twice the area of $\triangle ABC$ equals $|\underline{a} \times \underline{b}|$. Twice the area of $\triangle A_1B_1C_1$ equals

$$\begin{aligned} \mid (\underline{c}_1 - \underline{a}_1) \times (c_1 - \underline{b}_1) \mid &= \mid [\lambda\underline{a} + (1-2\lambda)\underline{b}] \times [(2\lambda-1)\underline{a} + (1-\lambda)\underline{b}] \mid \\ &= \mid 3\lambda^2 - 3\lambda + 1 || \underline{a} \times \underline{b} \mid \end{aligned}$$

The ratio of the area of $\triangle A_1B_1C_1$ to the area of $\triangle ABC$ is therefore $|3\lambda^2 - 3\lambda + 1|$, which is minimized when $\lambda = \dfrac{1}{2}$. Therefore A_1, B_1, C_1 are the midpoints of, respectively, the sides $\overline{BC}, \overline{CA}$ and $\overline{AB}$ of $\triangle ABC$.

Also solved by:

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