PROBLEM OF THE WEEK

Solution of Problem No. 13 (Spring 2007 Series)

Problem: Determine the polynomial $p(x)$ of degree up to 3 which minimizes

$$m = \max_{0\le x\le 1} \left| \cos 4\pi x - p(x) \right| .$$

Prove your answer.

Solution (by Pete Kornya, Faculty, Ivy Tech, Bloomington)

If $p(x)$ is the zero polynomial, then $m = \max_{0\le x\le 1} \left| \cos 4\pi x \right| = 1$. Now suppose that $p(x)$ is a polynomial of degree up to 3 such that $m \le 1$. We show that $p(x)$ must be the zero polynomial. Since $m \le 1$

$$p(0), p(0.5), p(1) \ge 0; \quad p(0.25), p(0.75) \le 0 \qquad (1)$$

Let Δ be the forward differencing operator $\Delta : p(x) \to p(x+0.25) - p(x)$. Then, since the degree of $p(x)$ is at most 3,

$$\begin{aligned} 0 &= \Delta^4 p(0) \\ &= p(1) - 4p(0.75) + 6p(0.5) - 4p(0.25) + p(0) \qquad (2) \\ &= \left[\, p(1) - p(0.75) \,\right] + 3 \left[\, p(0.5) - p(0.75) \,\right] + 3 \left[\, p(0.5) - p(0.25) \,\right] + \left[\, p(0) - p(0.25) \,\right] \end{aligned}$$

By (1) and the last line of (2), $p(1) = p(0.75) = p(0.5) = p(0.25) = p(0) = 0$. Since the degree of $p(x)$ is at most 3, and $p(x)$ has at least 5 zeros, it must be the zero polynomial. Therefore the required polynomial $p(x)$ is the zero polynomial.

Also solved by:

Mark Crawford (Waubonsee Community College instructor), Georges Ghosn (Quebec)