PROBLEM OF THE WEEK

Solution of Problem No. 14 (Spring 2007 Series)

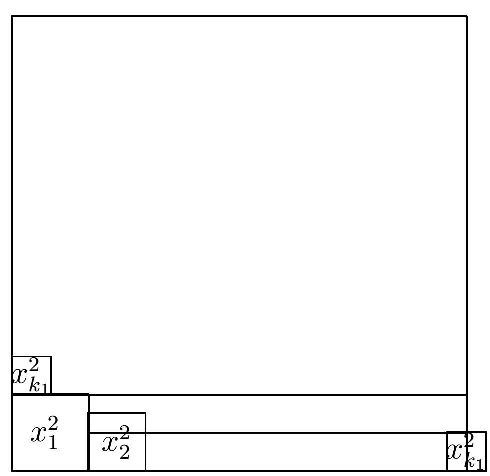
May 2, 2007

Problem: Any mumber of squares with total area less than or equal to 1/2 can be packed into the square of area 1.

Solution (by Prof. Harold Donnelly, Purdue)

Note that uncountable squares of positive areas will have total area ∞. So that there are countably many squares of positive areas. We may listed them in a non-increasing order,i.e. $x_1 \geq x_2 \geq \cdots$ with $\sum_{i=1}^{\infty} x_i^2 \leq 1/2$.

Consider again a box with base of length 1. Put the largest square into left hand corner and second largest next to it, etc., forming a horizontal row, until square no longer fit. Then move to horizontal row above and iterate, see the following diagram



Let the first one which can not be fitted in the first row be called $x_{k_1}^2$. Then we have the following inequality

$$(1 - x_1)x_{k_1} \leq \sum_{j=2}^{k_1} x_j^2.$$

Let the first one which can not be fitted in the second row be called $x_{k_2}^2$. Then we have the following inequality

$$(1 - x_1)x_{k_2} \leq \sum_{j=k_1+1}^{k_2} x_j^2$$

etc. Let $h = x_1 + x_{k_1} + \cdots$ be the total height of all rows. Then we have

$$h \leq x_1 + (1 - x_1)^{-1}(\sum_{j=2}^{\infty} x_j^2) \leq x_1 + (1 - x_1)^{-1}(1/2 - x_1^2) \leq 1.$$

The last step is the following elementary inequality

$$1/2 \leq x_1^2 + (1 - x_1)^2.$$

Also solved by: Georges Ghosn, Quebec.