PROBLEM OF THE WEEK
Solution of Problem No. 3 (Spring 2007 Series)

Problem: Find the last two decimal digits of 2007^{2007}. Computers not allowed. Show your work.

Solution (by Daniel Vacaru, Pitesti, Romania)

We know that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ (Newton's binomial theorem); from this fact, $2007^{2007} = (2000+7)^{2007} = \sum_{k=0}^{2007} \binom{2007}{k} 2000^{2007-k} 7^k$. From this fact, we deduce the last two digits of 2007^{2007}. These digits are the same with those of 7^{2007}. But we have

$$7^1 = 7$$
$$7^2 = 49$$
$$7^3 = 343 = \overline{.43}$$
$$7^4 = 2401 = \overline{..01}$$

By induction, we have $7^{4k+1} = \overline{...07}$, $7^{4k+2} = \overline{...49}$, $7^{4k+3} = \overline{..43}$, and $7^{4k} = \overline{..01}$ (because $7^4 = 2401$, and from the algorithm for multiplication). We have,
$2007 = 2004 + 3 = 4 \cdot 501 + 3$, and, consequently, the last two digits of 2007^{2007} are 43.

Also solved by:

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Graduates: Tom Engelsman (ECE), Thanh Duc Pham (IE)

Others: Manuel Barbero (New York), Mark Crawford (Waubonsee Community College instructor), Prithwijit De (Ireland), William DeVries (Warren Central HS, Indy), Ryan Dorow (Case Western Reserve U.), Sarah Friche–Moori (Warren Central HS, Indy), Georges

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