PROBLEM OF THE WEEK

Solution of Problem No. 4 (Spring 2007 Series)

February 27, 2007

Problem: A rod of length 1 is broken into four pieces. What is the probability that the four pieces are the sides of a trapezoid?

Solution (revised) (by the panel)

Let the (positive) lengths of the four pieces be x, y, z, and $w = 1 - x - y - z$. Claim: These are the lengths of the sides of a nondegenerate trapezoid $T \iff$ they are all $< 1/2$.

Proof. ($\Rightarrow$) If, say, $x \geq 1/2$, then $x \geq y + z + w$, hence T cannot exist.

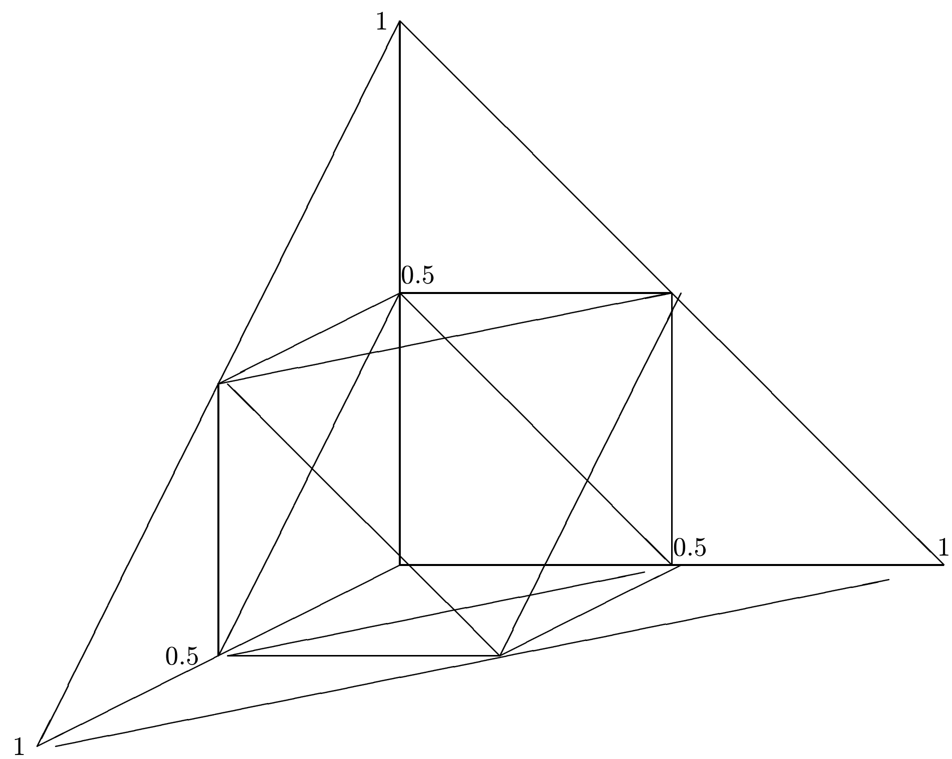
($\Leftarrow$) Three positive numbers a_1, a_2, a_3 are the lengths of the sides of a triangle if (and, of course, only if) the three triangle inequalities $a_i < a_j + a_k$ $(i, j, k$ distinct) all hold. Indeed, if, say, $a_1 \geq a_2 \geq a_3$, then two circles of radii a_2 and a_3 centered at the end points of a line segment of length a_1 must intersect.

Under the harmless assumption that $x \geq y \geq z \geq w$, we have

$$x - w < y + z, \quad y < x - w + z, \quad z < x - w + y;$$

so there is a triangle ABC with sides of respective lengths $x - w$, y and z. Extend the side AB by w units to a segment AD of length x, and let E be a point such that $BDEC$ is a parallelogram. Then $T := ADEC$ is a trapezoid with sides of length x, y, w, z. □

The meaning of the conditions of the statement can be illustrated by the following diagram where the four corners correspond to the four inequalities $x, y, z \geq 1/2, x + y + z \leq 1/2$. We have to cut off the four corners to get the (x, y, z) for which x, y, z, w are the lengths of the sides of a trapezoid ..



From the above we know that we have to cut off four small regular tetrahedra around the four vertices. Since each small tetrahedron has volume 1/8 of the total volume, we discard $4(1/8) = 1/2$ of the total volume. We conclude that the probability is $1 - 1/2 = 1/2$.

Also solved by:

<u>*Undergraduates :*</u> Alan Bernstein (Sr. ECE), Nathan Claus (Fr. Math).

<u>*Graduates :*</u> Tomek Czajka (CS), Tom Engelsman (ECE)

<u>*Others :*</u> Manuel Barbero (New York), Georges Ghosn (Quebec), Steven Landy (IUPUI, Physics)