PROBLEM OF THE WEEK

Solution of Problem No. 5 (Spring 2007 Series)

Problem: Determine all real $a > 0$ such that the series,

$$\sum_{n=1}^{\infty} a^{(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n})}$$

converges.

Solution (by Brad Lucier, Professor of Math, Purdue U)

Clearly, $a < 1$ for convergence, otherwise the terms don't tend to zero. Call the sum S. We have

$$\log n < \sum_{k=1}^{n} \frac{1}{k} \leq 1 + \log n, \qquad (1)$$

so

$$S \leq \sum_{n=1}^{\infty} a^{\log n} = \sum_{n=1}^{\infty} n^{\log a}$$

and the right hand side converges if $\log a < -1$, i.e., $a < e^{-1}$.

Similarly, using the other part of (1),

$$S > a \sum_{n=1}^{\infty} a^{\log n} = a \sum_{n=1}^{\infty} n^{\log a},$$

which diverges if $a \geq e^{-1}$

Also solved by:

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