PROBLEM OF THE WEEK

Solution of Problem No. 9 (Spring 2007 Series)

Problem: Let $f(x) = \dfrac{2x}{1+e^{2x}}$. Show that the n-th in derivative $f^{(n)}(0)$ is an integer for all $n \geq 0$.

Solution (by the Panel)

Given $f(x) = \dfrac{2x}{1+e^{2x}}$; let $g(x) = x$; $\quad h(x) = \dfrac{2}{1+e^{2x}}$. Then we have

$$\begin{aligned}
f(x) &= g(x) \cdot h(x) \\
f^{(1)}(x) &= g'(x) \cdot h(x) + h'(x) \cdot g(x) \\
f^{(2)}(x) &= g^{''}(x) \cdot h(x) + 2h'(x) \cdot g'(x) + h^{''}(x) \cdot g(x) \\
f^{(3)}(x) &= g^{'''}(x) \cdot h(x) + 3g^{''}(x) \cdot h'(x) + 3h^{''}(x) \cdot g'(x) + h^{'''}(x) \cdot g(x) \\
&\cdot
\end{aligned}$$

So, from induction we have,

$$f^{(n)} = \sum_{k=0}^{n} c_k^n h^{(k)}(x) \cdot g^{(n-k)}(x).$$

Therefore $f^{(n)}(0)$ is an integer if $c_k^n h^{(k)}(0) g^{(n-k)}(0)$'s are all integers. Note that c_k^n is a binomial coefficient which is an integer and $g^{(n-k)}(x) = x$ if $n-k=0$, $g^{(n-k)}(x) = 1$ if $n-k=1$, and $g^{(n-k)}(x) = 0$ if $n-k \geq 1$, hence $g^{(n-k)}(0)$'s are all integers. To solve our problem it suffices to show that $h^{(k)}(0)$ are all integers.

We claim that $h^{(k)}(x) = (-1)^k 2^{k+1} (1+e^{2x})^{-(k+1)} r_k(e^{2x})$ where $r_k(x)$ is a suitable polynomial with integer coefficients for $k = 0, 1, 2, \ldots$.

Proof of the claim: (1) For $k=0$, the claim is clear by taking $r_0(e^{2x}) = 1$. In general, let us assume that it is true for some k. Note that $r_k^{(1)}(e^{2x})$ is either 0 if the polynomial is a constant, or with all coefficients even. In any case we may take $r_k^{(1)}(e^{2x}) = 2s_k^{(1)}(e^{2x})$ with $s_k(x)$ a polynomial with integer coeffieients. Then we have

$$\begin{aligned}
h^{(k+1)}(x) &= (-1)^k 2^{k+1} (-(k+1))(1+e^{2x})^{-(k+2)} 2e^{2x} r_k(e^{2x}) + (-1)^k 2^{k+1} (1+e^{2x})^{-(k+1)} r_k^{(1)}(e^{2x}) \\
&= (-1)^{k+1} 2^{k+2} (1+e^{2x})^{-(k+2)} [(k+1)(e^{2x}) r_k(e^{2x}) - (1+e^{2x}) s_k(e^{2x})] \\
&= (-1)^{k+1} 2^{k+2} (1+e^{2x})^{-(k+2)} r_{k+1}(e^{2x}).
\end{aligned}$$

Now our claim is proved. It is easy to see that $h^{(k)}(0) = (-1)^k 2^{k+1} r_k(1)/2^{k+1}$ an integer.

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