

PROBLEM OF THE WEEK
Solution of Problem No. 1 (Spring 2008 Series)

Problem: Prove that, up to congruence, there are exactly three right triangles whose side lengths are integers while the area is twice the perimeter.

Solution (by Aviv Adler, Junior, College Prep. HS, CA)

Let a, b and $\sqrt{a^2 + b^2}$ be the sides of the right triangle then

$$\begin{aligned}\frac{1}{2}ab &= 2(a + b + \sqrt{a^2 + b^2}) \Leftrightarrow ab - 4a - 4b = 4\sqrt{a^2 + b^2} \\ \Rightarrow (ab - 4a - 4b)^2 &= 16(a^2 + b^2) \Rightarrow a^2b^2 - 8a^2b - 8ab^2 + 32ab = 0 \\ \Rightarrow ab - 8a - 8b + 64 &= 32 \Rightarrow (a - 8)(b - 8) = 32\end{aligned}$$

32 can be factored into integers in six ways:

$$(-1) \times (-32), \quad (-2) \times (-16), \quad (-4) \times (-8)$$

$$1 \times 32 \quad 2 \times 16 \quad 4 \times 8.$$

However, the first three would require $a \leq 0$ or $b \leq 0$, so they cannot work. Therefore, only the latter three can be used.

Adding 8 to both numbers in each of the last three factors we get that the sides of the required triangle can only be

$$(9, 40, 41), \quad (10, 24, 26) \quad \text{and} \quad (12, 16, 20).$$

Also solved by:

Undergraduates: Ramul Kumar (Sr. ECE), Hetong Li (Fr. Science), Nate Orlow (Jr. Math), John Joseph Steenberg (Sr. Math & Stat), Fan Zhang (So. CS)

Graduates: Abhishek Arora (ECE), Tom Engelsman (ECE), George Hassapis (Math), Jim Vaught (ECE)

Others: Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Youssef Chtaibi (PhD. France), Randin Divelbiss (Undergraduate, University of Wisconsin-Stevens)

Point), Nathan Faber (Sr., Case Western Reserve Univ.), Elie Ghosn (Montreal, Quebec), Pete Kornya (Faculty, Ivy Tech), Logeswaran Lajanugen (Highlands College, Hatton, Sri Lanka), Steven Landy (IUPUI Physics), Rajeev Malhotra (MD, Harvard Univ.), Graeme McRae (Palmdale CA), Chuck Ricks (Fort Wayne, IN), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago), Sahana Vasudevan (6th grade, Miller Middle School, CA), Bill Wolber Jr. (ITaP)

Note: Problem 14 from last semester was also solved by Randin Divelbiss, undergraduate at University of Wisconsin–Stevens Point, and Richard Divelbiss of Ferimilab.