PROBLEM OF THE WEEK
Solution of Problem No. 10 (Spring 2008 Series)

Problem: Let ABC be a (non–degenerate) triangle and a, b, c the lengths of the sides opposite A, B, C, respectively. Show that there is a triangle A', B', C' with corresponding sides $\sqrt{a}, \sqrt{b}, \sqrt{c}$. Show further that $\angle B'A'C' > \frac{1}{2}\angle BAC$.

Solution (by Brian Bradie, Professor, Christopher Newport U. VA)

Because ABC is a non–degenerate triangle, we have $a, b, c > 0$ and

$$a + b > c, \quad b + c > a \quad \text{and} \quad c + a > b. \tag{1}$$

With $a, b, c > 0$, it follows that

$$\sqrt{a+b} < \sqrt{a + 2\sqrt{ab} + b} = \sqrt{a} + \sqrt{b}. \tag{2}$$

Similarly,

$$\sqrt{b+c} < \sqrt{b} + \sqrt{c} \quad \text{and} \quad \sqrt{c+a} < \sqrt{c} + \sqrt{a}. \tag{3}$$

Combining (1), (2) and (3) yields

$$\begin{aligned} &\sqrt{a} + \sqrt{b} > \sqrt{a+b} > \sqrt{c}; \\ &\sqrt{b} + \sqrt{c} > \sqrt{b+c} > \sqrt{a}; \quad \text{and} \\ &\sqrt{c} + \sqrt{a} > \sqrt{c+a} > \sqrt{b}. \end{aligned}$$

From these last three inequalities, it follows that there exists a triangle $A'B'C'$ with corresponding sides $\sqrt{a}, \sqrt{b}, \sqrt{c}$. Now, by the Law of Cosines,

$$\cos(\angle BAC) = \frac{b^2 + c^2 - a^2}{2bc}.$$

Because $0 < \angle BAC < \pi$,

$$\begin{aligned} \cos\left(\frac{1}{2}\angle BAC\right) &= \sqrt{\frac{1 + \cos(\angle BAC)}{2}} = \frac{\sqrt{(b+c)^2 - a^2}}{2\sqrt{bc}} \\ &= \frac{\sqrt{(b+c-a)(b+c+a)}}{2\sqrt{bc}} \\ &> \frac{b+c-a}{2\sqrt{bc}} = \cos(\angle B'A'C'). \end{aligned}$$

Finally, as $\cos\theta$ is decreasing for $0 < \theta < \pi$, it follows that $\angle B'A'C' > \frac{1}{2}\angle BAC$.

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