PROBLEM OF THE WEEK

Solution of Problem No. 2 (Spring 2008 Series)

Problem: Assume $0 < q < 1$ and $f(x) = \dfrac{\sinh qx}{\sinh x}$. Show that f is monotone decreasing on $(0, \infty)$.

Solution (by George Hassapis, Math. Graduate student, Purdue)

Step 1: Consider the function $g(x) = q \tanh x - \tanh(qx)$, $x > 0$. We will prove that g is strictly decreasing and negative on $(0, +\infty)$. We have

$$g'(x) = q(\cosh x)^{-2} - q[\cosh(qx)]^{-2}, \text{ for all } x > 0.$$

Now cosh is strictly increasing on $(0, +\infty)$ and for all $x > 0$ we have $qx < x$, so $\cosh(qx) < \cosh x$ which implies $[\cosh(qx)]^2 < (\cosh x)^2$, since $0 < \cosh(qx) < \cosh x$. Thus $g'(x) < 0$ for all $x > 0$ i.e. g is strictly decreasing on $(0, +\infty)$. Therefore $g(x) < g(0) = 0$, for all $x > 0$.

Step 2: Now, the derivative of f on $(0, +\infty)$ is

$$f'(x) = \frac{q\cosh(qx)\sinh x - \sinh(qx)\cosh x}{(\sinh x)^2} = \cosh(qx)\cosh x \frac{g(x)}{(\sinh x)^2}$$

which is obviously negative on $(0, +\infty)$ since $\cosh(qx), \cosh x$, and $(\sinh x)^2$ are positive and $g(x)$ is negative on $(0, +\infty)$. Thus f is strictly decreasing on $(0, +\infty)$.

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