PROBLEM OF THE WEEK

Solution of Problem No. 3 (Spring 2008 Series)

Problem: Show that a real number q is rational if and only if there are three distinct integers, n_1, n_2, n_3, such that $q + n_1$, $q + n_2$, $q + n_3$ forms a geometric progression.

Solution (by Richard B. Eden, Math. Graduate student, Purdue)

Suppose q is a rational number. If $q = 0$, we can choose $n_1 = 1$, $n_2 = 2$, $n_3 = 4$. So now suppose $q = \frac{r}{s}$, not necessarily in lowest terms, where $r, s \in \mathbb{Z}$, $s \neq 0$ and $r \neq 0$. We can also assume $s \notin \{-1, -2\}$ since we can multiply r and s by the same constant.

Let $n_1 = 0$, $n_2 = r$, $n_3 = 2r + rs$. These three integers are distinct since $r \neq 0$ and $s \neq -1, -2$. In this case,

$$\begin{aligned} q + n_1 &= \frac{r}{s} + 0 = \frac{r}{s} \\ q + n_2 &= \frac{r}{s} + r = \frac{r}{s}(1 + s) \\ q + n_3 &= \frac{r}{s} + 2r + rs = \frac{r}{s}(1 + s)^2 \end{aligned}$$

really do form a geometric sequence.

Now suppose $q + n_1$, $q + n_2$, $q + n_3$ form a geometric sequence with n_1, n_2, n_3 distinct integers. This means $(q + n_2)^2 = (q + n_1)(q + n_3)$, which implies

$$(n_1 + n_3 - 2n_2)q = n_2^2 - n_1 n_3.$$

If $(n_1 + n_3 - 2n_2) = 0$, so $n_2 = \dfrac{n_1 + n_3}{2}$, we can write $n_1 = n_2 - d$ and $n_3 = n_2 + d$ for some $d \in \mathbb{Z}$. We then have, from the above equation,

$$0 = n_2^2 - n_1 n_3 = n_2^2 - (n_2 - d)(n_2 + d) = d^2$$

so $d = 0$ and $n_1 = n_2 = n_3$. However, n_1, n_2, n_3 are all distinct. Therefore, $n_1 + n_3 - 2n_2 \neq 0$ and

$$q = \frac{n_2^2 - n_1 n_3}{n_1 + n_3 - 2n_2}$$

which is a rational number.

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