PROBLEM OF THE WEEK

Solution of Problem No. 5 (Spring 2008 Series)

Problem: For given positive integers a and n, show that there is a positive integer b such that

$$(\sqrt{a}-\sqrt{a-1})^n = \sqrt{b}-\sqrt{b-1}.$$

Solution (by Elie Ghosn, Montreal, Quebec)

We have:

$$2\sqrt{b} = (\sqrt{b}+\sqrt{b-1}) + (\sqrt{b}-\sqrt{b-1}) = \frac{1}{\sqrt{b}-\sqrt{b-1}} + \sqrt{b}-\sqrt{b-1}$$
$$= (\sqrt{a}+\sqrt{a-1})^n + (\sqrt{a}-\sqrt{a-1})^n$$

$\Rightarrow b = \frac{1}{4}\left[(\sqrt{a}+\sqrt{a-1})^n + (\sqrt{a}-\sqrt{a-1})^n\right]^2.$

We can easily verify that this value is a solution of the equation:

Indeed

$b-1 = \frac{1}{4}\left[(\sqrt{a}+\sqrt{a-1})^n + (\sqrt{a}-\sqrt{a-1})^n\right]^2 - 1 = \frac{1}{4}\left[(\sqrt{a}+\sqrt{a-1})^n - (\sqrt{a}-\sqrt{a-1})^n\right]^2$

$\Rightarrow \sqrt{b}-\sqrt{b-1} = (\sqrt{a}-\sqrt{a-1})^n.$

By using the binomial formula, we have:

$$b = \left[\frac{1}{2}\sum_{k=0}^{n}(1+(-1)^k)\binom{n}{k}(\sqrt{a-1})^k(\sqrt{a})^{n-k}\right]^2.$$

Therefore,

$$b = \begin{cases} \left(\sum\limits_{k=0}^{p}\binom{2p}{2k}(a-1)^k a^{p-k}\right)^2, & \text{if } n = 2p \\ \left(\sum\limits_{k=0}^{p}\binom{2p+1}{2k}(a-1)^k a^{p-k}\right)^2 \cdot a, & \text{if } n = 2p+1. \end{cases}$$

Both are integers!

Also solved by:

Graduates: George Hassapis (Math)

Others: Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Hoan Duong (San Antonio College), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago)