PROBLEM OF THE WEEK

Solution of Problem No. 6 (Spring 2008 Series)

Problem: Show that the sequence $\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}$, $n = 1, 2, \ldots$ is decreasing.

Solution (by Jeremy Rocke, Freshman, Christopher Newport University)

We will show that the sequence $\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}$ is decreasing by proving that the function $f(x) = \left(1+\frac{1}{x}\right)^{x+\frac{1}{2}}$ is decreasing on $(0, \infty)$.

$$f(x) = e^{\ln\left(1+\frac{1}{x}\right)^{x+\frac{1}{2}}} = e^{\left(x+\frac{1}{2}\right)\ln\left(1+\frac{1}{x}\right)}$$

$$f'(x) = e^{\left(x+\frac{1}{2}\right)\ln\left(1+\frac{1}{x}\right)}\left[\left(\frac{-x^{-2}}{1+\frac{1}{x}}\right)\cdot\left(x+\frac{1}{2}\right)+\ln\left(1+\frac{1}{x}\right)\right]$$

$$f'(x) = e^{\left(x+\frac{1}{2}\right)\ln\left(1+\frac{1}{x}\right)}\left[\frac{-x^{-1}-\frac{1}{2}x^{-2}}{1+\frac{1}{x}}+\ln\left(1+\frac{1}{x}\right)\right]$$

$$f'(x) = e^{\left(x+\frac{1}{2}\right)\ln\left(1+\frac{1}{x}\right)}\left[\frac{(-2x-1)}{2x(x+1)}+\ln\left(1+\frac{1}{x}\right)\right]$$

We know that $e^{\left(x+\frac{1}{2}\right)\ln\left(1+\frac{1}{x}\right)}$ is positive so we will be looking at the other factor. Let $g(x) = \frac{(-2x-1)}{2x^2+2x} + \ln\left(1+\frac{1}{x}\right)$. Now we take the derivative of $g(x)$ and we get

$$g'(x) = \frac{-\left(-x-\frac{1}{2}\right)(2x+1)}{(x^2+x)^2} + \frac{-1}{x^2+x} - \frac{\frac{1}{x^2}}{1+\frac{1}{x}}$$

$$g'(x) = \frac{\frac{1}{2}}{(x^2+x)^2}$$

Clearly $g'(x)$ is positive on $(0, \infty)$ which implies that $g(x)$ is increasing on $(0, \infty)$. But $\lim_{x\to\infty} g(x) = \lim_{x\to\infty}\left[\frac{(-2x-1)}{2x^2+2x}+\ln\left(1+\frac{1}{x}\right)\right] = 0$. So as x gets big, $g(x)$ increases to 0. The only way that can happen is if $g(x)$ is negative on $(0, \infty)$. Thus $f'(x) = e^{\left(x+\frac{1}{2}\right)\ln\left(1+\frac{1}{x}\right)} g(x)$ is negative on $(0, \infty)$ and so $f(x)$ is decreasing on $(0, \infty)$. In particular, the sequence $\left(1 + \dfrac{1}{n}\right)^{n+\frac{1}{2}}$, $n = 1, 2, \ldots$ is decreasing.

Also solved by:

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