PROBLEM OF THE WEEK

Solution of Problem No. 1 (Spring 2009 Series)

Problem: Find a formula for the determinant of the 2009 $\times$ 2009 matrix whose (i,j)–entry is $\delta_{ij} + x_i x_j$. Your answer must be justified without the use of computers. $\left(\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}.\right)$

Solution (by Yansong Xu, Brandon, FL)

Claim: $\det(\delta_{ij} + x_i x_j)_{n\times n} = 1 + \sum_{i=1}^{n} x_i^2$. Proof by induction. For $n = 1$, the claim is trivially true. Suppose the claim is true for $n-1$. For $i = 1, \dots, n-1$, subtract the n–row multiplied by $\frac{x_i x_n}{1+x_n^2}$, from the i–row.

$$\det\left(\delta_{ij} + x_i x_j\right)_{n\times n} = \det \left[\begin{array}{c|c} \left(\delta_{ij} + \frac{x_i}{\sqrt{1+x_n^2}} \cdot \frac{x_j}{\sqrt{1+x_n^2}}\right)_{(n-1)\times(n-1)} & 0 \\ \cdots\cdots\cdots & \cdots \\ * & 1+x_n^2 \end{array}\right]$$

$$\begin{aligned} &= \det\left(\delta_{ij} + \frac{x_i}{\sqrt{1+x_n^2}} \cdot \frac{x_j}{\sqrt{1+x_n^2}}\right)_{(n-1)\times(n-1)} \cdot (1+x_n^2) \\ &= \left(1 + \sum_{i=1}^{n-1} \left(\frac{x_i}{\sqrt{1+x_n^2}}\right)^2\right) \cdot (1+x_n^2) \\ &= 1 + \sum_{i=1}^{n} x_i^2. \end{aligned}$$

Also completely or partially solved by:

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