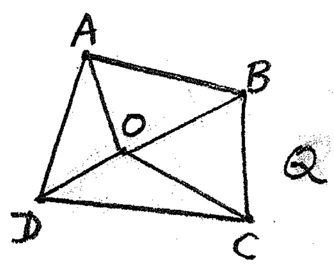
PROBLEM OF THE WEEK
Solution of Problem No. 10 (Spring 2009 Series)

Problem: Let Q be a convex quadrilateral each of whose sides has length at most 20. Show that if O is an arbitrary interior point of Q, then at least one of the vertices of Q has distance less than 15 from O.

Solution (by Xingyi Qin, Sr., Actuarial Science, Purdue University)

Suppose all vertices of Q have distance of at least 15 from O. Use the Law of cosines:



$$\cos\angle AOB = \frac{\overline{AO}^2 + \overline{BO}^2 - \overline{AB}^2}{2\cdot\overline{AO}\cdot\overline{BO}} \geq \frac{15^2+15^2-20^2}{2\cdot 15\cdot 15} = \frac{1}{9}$$

$$\Rightarrow \angle AOB \leq \arccos\frac{1}{9} < \frac{\pi}{2}$$

For the same reason,

$$\angle BOC < \frac{\pi}{2}, \quad \angle COD < \frac{\pi}{2}, \quad \angle DOA < \frac{\pi}{2}$$

$$\Rightarrow \angle AOB + \angle BOC + \angle COD + \angle DOA < \frac{\pi}{2}\cdot 4 = 2\pi.$$

This is a contradiction. So the hypothesis is not valid, which means at least one of the vertices of Q has distance less than 15 from O.

The problem was also solved by:

Undergraduates: Andy Bohn (Jr. Phys)

Graduates: Richard Eden (Math), Phuong Thanh Tran (ECE)

Others: Haonan Chen (China), Gruian Cornel (IT, Romania), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Tigran Hakobyan (Armenia), Jeffery Hein (CS & Math, Purdue Univ. Calumet), Steven Landy (IUPUI Physics staff), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel)