PROBLEM OF THE WEEK

Solution of Problem No. 14 (Spring 2009 Series)

Problem: A number c, $0 < c \leq 1$, is called a <u>chord–number</u> if for every continuous function $f : [0,1] \to \mathbb{R}$ with $f(0) = f(1) = 0$, there is a point x_0 in $[0, 1-c]$ such that $f(x_0) = f(x_0 + c)$. Show that $\{\frac{1}{n} : n = 1, 2, \ldots, \}$ are the only chord–numbers.

Solution (by Sorin Rubinstein, TAU faculty, Israel)

Assume that for some positive integer n the number $\dfrac{1}{n}$ is not a chord number, and let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1) = 0$ and $f\left(x+\dfrac{1}{n}\right) - f(x) \neq 0$ for every $x \in \left[0, 1-\dfrac{1}{n}\right]$. Then the function $g(x) := f\left(x+\dfrac{1}{n}\right) - f(x)$ is either strictly positive or strictly negative on $\left[0, 1-\dfrac{1}{n}\right]$. We assume without lost of generality that $g(x) > 0$ on this interval. Then $g\left(\dfrac{k}{n}\right) = f\left(\dfrac{k+1}{n}\right) - f\left(\dfrac{k}{n}\right) > 0$ for $k = 0, 1, 2, \ldots, n-1$. Therefore $\sum\limits_{k=0}^{n-1} g\left(\dfrac{k}{n}\right) > 0$. On the other hand $\sum\limits_{k=0}^{n-1} g\left(\dfrac{k}{n}\right)$ is a telescoping sum and $\sum\limits_{k=0}^{n-1} g\left(\dfrac{k}{n}\right) = \sum\limits_{k=0}^{n-1} \left(f\left(\dfrac{k+1}{n}\right) - f\left(\dfrac{k}{n}\right)\right) = f(1) - f(0) = 0$ which is a contradiction. Thus $\dfrac{1}{n}$ must be a chord number for $n = 1, 2, 3 \ldots$. Now, let c, $0 < c < 1$ be a number such that $c \neq \frac{1}{n}$ for $n = 1, 2, 3 \ldots$. For every $x \in \mathbb{R}$ let $h(x)$ be the distance from x to the nearest integer. This is a continuous function. Define the function $f : [0,1] \to \mathbb{R}$ by:

$$f(x) = h\left(\frac{x}{c}\right) - xh\left(\frac{1}{c}\right)$$

Then f is continuous, $f(0) = f(1) = 0$ and, since $h\left(\dfrac{x+c}{c}\right) = h\left(\dfrac{x}{c} + 1\right) = h\left(\dfrac{x}{c}\right)$,

$$f(x+c) = h\left(\frac{x+c}{c}\right) - (x+c)h\left(\frac{1}{c}\right) = h\left(\frac{x}{c}\right) - xh\left(\frac{1}{c}\right) - ch\left(\frac{1}{c}\right) = f(x) - ch\left(\frac{1}{c}\right).$$

Since $\dfrac{1}{c}$ is not an integer $h\left(\dfrac{1}{c}\right) \neq 0$ and, therefore, $f(x+c) \neq f(x)$ for every x in $[0, 1-c]$. Thus c is not a chord number.

The problem was also solved by:

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