PROBLEM OF THE WEEK

Solution of Problem No. 8 (Spring 2009 Series)

Problem: Let $A_1, \ldots, A_n$ be finite sets and let a_{ij} be the cardinality of $A_i \cap A_j$. Show that the $n \times n$ matrix (a_{ij}) is positive semidefinite. (In other words, show that $\sum_{i,j=1}^{n} a_{ij} x_i x_j \geq 0$ for all real numbers $x_1, \ldots, x_n$.)

Solution (by Neacsu Adrian, Romania)

Let $B = A_1 \cup \cdots \cup A_n = \{b_1, \ldots, b_q\}$

Define C in $M_{nq}(R)$ such that $C_{ij} = 1$, if b_j in A_i and $C_{ij} = 0$, if b_j not in A_i, i from 1 to n, j from 1 to q. Then $CC^t = A$.

If $X = (x_1 \ldots x_n)$ in $M_{1n}(R)$ corresponding sum can be written

$$XAX^t = XCC^tX^t = (XC)(XC)^t = DD^t$$

where $D = XC$ in $M_{1q}(R)$, so $DD^t = d_1^2 + \cdots + d_q^2 \geq 0$.

The problem was also solved by:

<u>Others</u>: Brian Bradie (Christopher Newport U. VA), Randin Divelbiss (University of Wisconsin–Wausau), John Hyde (Hoover, AL), Steven Landy (IUPUI Physics staff), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Sorin Rubinstein (TAU faculty, Israel), Yansong Xu (Brandon, FL)