PROBLEM OF THE WEEK

Solution of Problem No. 9 (Spring 2009 Series)

Problem: If n is a given positive integer, how many solutions (x, y) does

$$\frac{1}{n} = \frac{1}{x} + \frac{1}{y}$$

have with x and y unequal positive integers?

Solution (by Gruian Cornel, IT, Romania)

If (x, y) is a solution for $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{n}$, clearly $\min(x, y) > n$. If not, say $\min(x, y) = x \leq n$ then $\frac{1}{n} = \frac{1}{x} + \frac{1}{y} > \frac{1}{n}$, contradiction. So $x > n$ and $y > n$. We write the equation as $n(x + y) = xy$ or $(x - n)(y - n) = n^2$ and for r positive $r|n^2$, the solutions are given by $x - n = r$ and $y - n = \dfrac{n^2}{r}$ or $x = n + r$ and $y = n + \dfrac{n^2}{r}$. The only case when $x = y$ is when $r = \dfrac{n^2}{r}$ or $r = n$ and the numbers of solutions (x, y) with $x \neq y$ is $d(n^2) - 1$ where $d(n^2)$ is the number of divisors of n^2, $d(n^2) = (2q_1 + 1)(2q_2 + 1) \ldots (2q_n + 1)$, where $n = p_1^{q_1} p_2^{q_2} \ldots p_m^{q_m}$ is the prime factorization of n.

The problem was also solved by:

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