PROBLEM OF THE WEEK

Solution of Problem No. 10 (Spring 2010 Series)

Problem: Prove that, if x and y are positive irrationals such that

$$\frac{1}{x} + \frac{1}{y} = 1,$$

then the sequences $[x], [2x], [3x], \ldots, [y], [2y], [3y], \ldots$ together include every positive integer exactly once.

Note: $[u]$ denotes the largest integer n satisfying $n \leq u$.

Solution (by Zhengpeng Wu, Tsinghua University, China)

First, we prove there are no positive integers m_0, n_0, satisfying $[m_0 x] = [n_0 y]$. Otherwise, we let $k = [m_0 x] = [n_0 y]$. Then we have $k < m_0 x < k+1$, $k < n_0 y < k+1$. Because x, y are irrational, there is no equality. Then we have

$$\frac{m_0}{k+1} < \frac{1}{x} < \frac{m_0}{k} \quad \text{and} \quad \frac{n_0}{k+1} < \frac{1}{y} < \frac{n_0}{k} \Rightarrow \frac{m_0+n_0}{k+1} < 1 < \frac{m_0+n_0}{k} \Rightarrow k < m_0+n_0 < k+1.$$

But there is no integer between k and $k+1$. So we get a contradiction.

Second, we prove $\{[mx]\}$ and $\{[ny]\}$ cover all positive integers. It is impossible that $x > 2, y > 2$, so we suppose $2 > x > 1$, $y > 1$ without loss of generality. Then the steps in $\{[mx]\}$ are 1 or 2. Then we prove $\{[ny]\}$ fills the gaps in $\{[mx]\}$ when the step is 2. Suppose $k < m_0 x < k+1$, $k+2 < (m_0+1)x < k+3$. Then

$$\begin{aligned}
&\frac{m_0}{k+1} < \frac{1}{x} < \frac{m_0}{k}, \quad \frac{m_0+1}{k+3} < \frac{1}{x} < \frac{m_0+1}{k+2} \\
&\Rightarrow \frac{k+1}{k+1-m_0} < y < \frac{k}{k-m_0} \quad \text{and} \quad \frac{k+3}{k+2-m_0} < y < \frac{k+2}{k+1-m_0} \\
&\Rightarrow k+1 < (k+1-m_0)y < k+2 \Rightarrow [(k+1-m_0)y] = k+1.
\end{aligned}$$

The gap in $\{[mx]\}$ is filled.

The problem was also solved by:

Undergraduates: Kilian Cooley (Fr.), Yixin Wang (Fr.)

Graduates: Richard Eden (Math), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Siavash Ameli (Grad. student, Toosi Univ. of Tech, Iran), Jim Carrubba (Science teacher, IL), Gruian Cornel (IT, Romania), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Wei-hsiang Lien (Research assistant, National Chiao-Tung Univ., Taiwan), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.) Mark Sellke (Klondike Middle School, IN), Steve Spindler (Chicago), Hyung-bin Youk (MD. Korea)