PROBLEM OF THE WEEK
Solution of Problem No. 11 (Spring 2010 Series)

Problem: Show that

$$\int_0^1 x^x\,dx = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-n}.$$

Solution (by Youness Oumzil, Lycée Michel Montaigne, France)

$$\int_0^1 x^x\,dx = \int_0^1 e^{x\ln(x)}dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(x\ln(x))^n}{n!}dx.$$

Since $\forall x \in [0,1], |x\ln(x)| \le 1$,
$\forall x \in [0,1]$, $\forall n$,

$$\frac{|x\ln(x)|^n}{n!} \le \frac{1}{n!}.$$

This shows that series of functions $\sum\limits_{n\ge 0} \dfrac{(x\ln(x))^n}{n!}$ converges uniformly on the interval $[0,1]$.
We can then exchange the order of the integral and the sum:

$$\int_0^1 \sum_{n=0}^{\infty} \frac{(x\ln(x))^n}{n!}dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(x\ln(x))^n}{n!}dx.$$

Let's consider the notation:

$$I_n = \int_0^1 \frac{(x\ln(x))^n}{n!}\,dx.$$

Let's show $\forall k \le n,\ I_n = \dfrac{(-1)^k}{(n+1)^k} \displaystyle\int_0^1 \frac{x^n \cdot (\ln(x))^{n-k}}{(n-k)!}\,dx$. Let's call this statement $P(k)$ and use mathematical induction:

@<u>for $k = 0$:</u> It is correct because it is the definition of I_n.

Assuming that $P(k)$ holds and $k < n$ we have:

$$I_n = \frac{(-1)^k}{(n+1)^k}\left[\frac{x^{n+1}\cdot(\ln(x))^{n-k}}{(n-k)!(n+1)}\right]_0^1 - \frac{(-1)^k}{(n+1)^k}\int_0^1 \frac{x^{n+1}\cdot(n-k)\left(\frac{1}{x}\right)\cdot(\ln(x))^{n-k-1}}{(n-k)!(n+1)}\,dx.$$

So

$$I_n = 0 - \frac{(-1)^k}{(n+1)^k} \int_0^1 \frac{x^n \cdot (\ln(x))^{n-k-1}}{(n-k-1)!(n+1)}\, dx = \frac{(-1)^{k+1}}{(n+1)^{k+1}} \int_0^1 \frac{x^n (\ln(x))^{n-k-1}}{(n-k-1)!}\, dx$$

By $P(n)$, $I_n = \dfrac{(-1)^n}{(n+1)^n} \displaystyle\int_0^1 x^n dx = \dfrac{(-1)^n}{(n+1)^{n+1}} = \dfrac{(-1)^{n+2}}{(n+1)^{n+1}}$. Then

$$\int_0^1 x^x dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(x\ln(x))^n}{n!} dx = \sum_{n=0}^{\infty} I_n = \sum_{n=0}^{\infty} \frac{(-1)^{n+2}}{(n+1)^{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^n}.$$

The problem was also solved by:

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