PROBLEM OF THE WEEK

Solution of Problem No. 13 (Spring 2010 Series)

Problem: Show that for $0 < \varepsilon < 1$ the expression $(x+1)^n(x^2-(2-\varepsilon)x+1)$ is a polynomial with strictly positive coefficients if n is sufficiently large. For $\varepsilon = 10^{-3}$ find the smallest possible n.

Solution (by Gruian Cornel, IT, Romania)

Let $p(x) = (x+1)^n(x^2-(2-\varepsilon)x+1) = \sum_{k=0}^{n+2} a_k x^{n+2-k}$ where $a_0 = a_{n+2} = 1$, $a_1 = a_{n+1} = n-(2-\varepsilon)$ and $a_{k+2} = \binom{n}{k+2} - (2-\varepsilon)\binom{n}{k+1} + \binom{n}{k}$ for $k = 0, 1, \ldots, n-2$. For $a_j > 0$, we have the conditions $\frac{(n-k)(n-k-1)}{(k+1)(k+2)} - (2-\varepsilon)\frac{n-k}{k+1} + 1 > 0$, or $(n+1)\left(\frac{1}{k+2} + \frac{1}{n-k}\right) > 4 - \varepsilon$ where $k = 0, 1, \ldots, n-2$. We need that (1): $(n+1)\min\left\{\left(\frac{1}{k+2} + \frac{1}{n-k}\right) : k = 0, 1, \ldots, n-2\right\} > 4 - \varepsilon$. Consider $f : [0, n-2] \to (0, \infty)$, $f(x) = \frac{1}{x+2} + \frac{1}{n-x}$, $f'(x) = \frac{(n+2)(2x-(n-2))}{(x+2)^2(n-x)^2}$, $f'(x) < 0$ on $\left[0, \frac{n-2}{2}\right)$, $f'(x) > 0$ on $\left(\frac{n-2}{2}, n-2\right]$, $\frac{n-2}{2}$ is a minimum point for f and $f\left(\frac{n-2}{2}\right) = \frac{4}{n+2}$. Hence if $\frac{4(n+1)}{n+2} > 4 - \varepsilon$, or $n > \frac{4}{\varepsilon} - 2$ then $a_j > 0$ for $j = 0, 1, \ldots, n+2$. Now we inspect the cases:

1) For n even, $n = 2m$ then $\min\left\{\left[\frac{1}{k+2} + \frac{1}{n-k}\right] : k = 0, \ldots, n-2\right\} = f(m-1) = \frac{2}{m+1}$ and (1) becomes $\frac{2(2m+1)}{m+1} > 4 - \varepsilon$, or $m > \frac{2}{\varepsilon} - 1$. For $\varepsilon = 10^{-3}$, $m_{\min} = 2 \cdot 10^3$ and $n_{\min} = 4000$.
2) For n odd, $n = 2m+1$ then $\min\left\{\left[\frac{1}{k+2} + \frac{1}{n-k}\right] : k = 0, \ldots, n-2\right\} = f(m) = f(m-1) = \frac{1}{m+1} + \frac{1}{m+2}$ and (1) becomes $\frac{2(2m+3)}{m+2} > 4 - \varepsilon$, or $m > \frac{2}{\varepsilon} - 2$. For $\varepsilon = 10^{-3}$, $m_{\min} = 2 \cdot 10^3 - 1$ and $n_{\min} = 4000 - 1 = 3999$.

Hence for $\varepsilon = 10^{-3}$ the smallest possible n is 3999.

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