PROBLEM OF THE WEEK

Solution of Problem No. 14 (Spring 2010 Series)

Problem: A sequence $a_0, a_1, a_2, \ldots$ of real numbers satisfies

(1) $0 \le a_0 \le 1$

and

(2) $a_{n+1} = 4a_n^3 - 6a_n^2 + a_n + 1 \qquad (n = 0, 1, 2, \ldots).$

Given that $\lim_{n\to\infty} a_n$ exists, find (with proof) the possible value(s) of a_0.

Solution (by Craig Schroeder, Ph.D. student, Stanford University)

Let $f(x) = 4x^3 - 6x^2 + x + 1$. Let a be such a limit. Then, $a = f(a)$. This has three solutions: $a = \frac{1}{2}$, $a = \frac{1}{2} \pm \frac{1}{2}\sqrt{3}$.

Let $r = \frac{1}{2} - \frac{2}{9}\sqrt{6}$ and $s = \frac{1}{2} + \frac{2}{9}\sqrt{6}$. Consider the interval $I = [r, s]$. Iteration starts in this interval, since $[0, 1] \subset I$. The extreme values of f occur at the endpoints or at local extrema. $f(r) = \frac{1}{2} + \frac{44}{243}\sqrt{6} \in I$ and $f(s) = \frac{1}{2} - \frac{44}{243}\sqrt{6} \in I$. $f'(x) = 12x^2 - 12x + 1$, so the critical points are $c\pm = \frac{1}{2} \pm \frac{1}{6}\sqrt{6}$, so that $f(c_-) = s$ and $f(c_+) = r$. Thus, $f(I) = I$. Since $\frac{1}{2} \pm \frac{1}{2}\sqrt{3} \notin I$, no valid starting point can converge to those values. Thus, any sequence that converges must converge to $\frac{1}{2}$.

The initial value $a_0 = \frac{1}{2}$ leads trivially to a constant sequence that converges. The other two solutions to $f(x) = \frac{1}{2}$ lie outside I. The other possibility is that the sequence converges to $\frac{1}{2}$ without actually obtaining that value. Let $a_n = \frac{1}{2} + \epsilon$, so that $a_{n+1} = \frac{1}{2} - 2\epsilon + 4\epsilon^3$. Assume that $|\epsilon| < \frac{1}{4}$, so that $|\frac{1}{2} - a_{n+1}| = 2|\epsilon||1 - 2\epsilon^2| > \frac{7}{4}|\epsilon| > |\epsilon|$. Since the sequence diverges from $\frac{1}{2}$, there are no other converging sequences. The only possible starting value is $a_0 = \frac{1}{2}$.

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