PROBLEM OF THE WEEK

Solution of Problem No. 4 (Spring 2010 Series)

Problem: Six circles, not necessarily of the same size, are drawn in a plane. Suppose that there exists some point which lies (strictly) inside all of the circles. Prove that the center of at least one of the circles lies (strictly) inside another of them.

Solution (by Steve Spindler, Chicago)

Denote by P the point which lies in the interior of the 6 circles. Label the centers of the circles as C_i, $i = 1$ to 6, where the ordering is clockwise about P. Let r_i denote the radius of the circle centered at C_i. Assume by way of contradiction that none of the C_i lie in the interior of any of the other 5 circles.

Consider the 6 triangles C_iPC_{i+1} (where $C_7 \stackrel{\text{def}}{=} C_1$). $|C_iC_{i+1}| \geq \max\{r_i, r_{i+1}\}$, since C_i does not lie in the interior of C_{i+1}, and vice–versa. Similarly, $|C_iP| < r_i$ and $|C_{i+1}P| < r_{i+1}$ since P lies in the interior of both circles. Thus, $|C_iC_{i+1}| > \max\{|C_iP|, |C_{i+1}P|\}$. Since side C_iC_{i+1} is strictly larger than the other two sides of triangle C_iPC_{i+1}, angle $\angle C_iPC_{i+1}$ is greater than the other two angles. Since the angles of any triangle add up to $180°$, it follows that $\angle C_iPC_{i+1} > 60°$. Thus the sum of all 6 angles is strictly greater than $6 \times 60° = 360°$. But these 6 angles must add up to exactly $360°$. This contradiction proves the result.

The problem was also solved by:

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