PROBLEM OF THE WEEK
Solution of Problem No. 5 (Spring 2010 Series)

Problem: Let R be a given positive real number, and k any fixed positive integer. Show that there are at most finitely many k–tuples of positive integers $a_1, \ldots, a_k$ such that

$$R = \frac{1}{a_1} + \cdots + \frac{1}{a_k}.$$

Solution (by Eric Haengel, Sophmore, Math & Physics, Purdue University)

Proceed by induction:
$k = 1 : R = \dfrac{1}{a_1} \Rightarrow a_1 = R^{-1}$. There is only one choice for a_1.

$k > 1 : R = \dfrac{1}{a_1} + \cdots + \dfrac{1}{a_n}$. Without loss of generality, assume that $a_n \leq a_j \quad \forall j$. Then $\dfrac{1}{a_n} \geq \dfrac{1}{a_j} \quad \forall j$. Therefore $R \leq \dfrac{k}{a_n} \quad \Rightarrow \quad a_n \leq \dfrac{k}{R}$. Hence for the smallest element in the k–tuples, there are only finitely many choices, as the elements are positive integers.

Now consider $R' = R - \dfrac{1}{a_n} = \dfrac{1}{b_1} + \cdots + \dfrac{1}{b_{n-1}}$. For each choice of a_n there are, by the induction hypothesis, only finitely many choices for $b_1, \ldots, b_{n-1}$. Since there are only finitely many choices for a_n, there are only finitely many k–tuples $(a_1, a_2, \ldots, a_n)$ with $a_1, \ldots, a_n$ positive integers such that $R = \dfrac{1}{a_1} + \cdots + \dfrac{1}{a_n}$.

The problem was also solved by:

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