PROBLEM OF THE WEEK

Solution of Problem No. 6 (Spring 2010 Series)

Problem: Let a, b, c be the side–lengths of a triangle. Show that

$$(a+b-c)^a(b+c-a)^b(c+a-b)^c \leq a^a b^b c^c$$

with "=" if and only if the triangle is equilateral.

Hint: Let $r = a+b-c$, $s = b+c-a$, and $t = c+a-b$, and generalize the problem by identifying the essential properties of r, s, and t.

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We received some very different solutions (and also a nice correct proof of a different but related inequality). Below are two solutions, one of which carries out the hint. The intended interpretation of the hint was to prove

$$r^a s^b t^c \leq a^a b^b c^c \quad \text{if } r, s, t > 0 \quad \text{and} \quad r+s+t = a+b+c,$$

with equality if and only if $r = a, s = b, t = c$.

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Solution #1 (by Steven Landy, IUPUI Physics staff)

We are to show

$$(a+b-c)^a(b+c-a)^b(c+a-b)^c \leq a^a b^b c^c \qquad [1]$$

which is the same as

$$\frac{a}{a+b+c}\log\left(\frac{a+b-c}{a}\right)+\frac{b}{a+b+c}\log\left(\frac{b+c-a}{b}\right)+\frac{c}{a+b+c}\log\left(\frac{c+a-b}{c}\right) \leq 0 \qquad [2]$$

By the concavity of the log function we have that the left side of [2] is $\leq$

$$\log\left(\frac{a}{a+b+c}\left(\frac{a+b-c}{a}\right)+\frac{b}{a+b+c}\left(\frac{b+c-a}{b}\right)+\frac{c}{a+b+c}\left(\frac{c+a-b}{c}\right)\right) \qquad [3]$$

with equality iff $a = b = c$. But [3] $= 0$. Thus [2] is true and so also is [1].

Solution #2 (by Elie Ghosn, Montreal, Quebec)

$r = a + b - c$; $s = b + c - a$ and $t = c + a - b$ are obviously positive numbers.
Consider the ratio:

$$M = \frac{r^a s^b t^c}{a^a b^b c^c} = e^{a\ln(\frac{r}{a}) + b\ln(\frac{s}{b}) + c\ln(\frac{t}{c})}$$

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From the inequality $\ln x \leq x - 1$ which is valid for all positive numbers x, and for which the equality holds for $x = 1$ only, we deduce:

$$M \leq e^{a(\frac{r}{a}-1)+b(\frac{s}{b}-1)+c(\frac{t}{c}-1)} = e^0 = 1$$

and the equality holds iff $\frac{r}{a} = 1$, $\frac{s}{b} = 1$ and $\frac{t}{c} = 1$, hence $a = b = c$, and therefore the triangle is equilateral.

The problem was also solved by:

Undergraduates: Kilian Cooley (Fr.), Yixin Wang (Fr.)

Graduates: Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Kevin Laster (Indianapolis, IN), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.)