PROBLEM OF THE WEEK

Solution of Problem No. 7 (Spring 2010 Series)

Problem: If m, n are positive integers such that

$$(m+1)^3 - m^3 = n^2,$$

show that

$$n = k^2 + (k+1)^2$$

for some integer k.

Solution (by Richard Eden, Graduate student, Math, Purdue University)

Since $n^2 = (m+1)^3 - m^3 = 3m^2 + 3m + 1$, then

$$4n^2 - 1 = 12m^2 + 12m + 3 = 3(2m+1)^2.$$

If p is prime and $p|(2m+1)^2$, then $p|2n+1$ or $p|2n-1$, but p cannot divide both $2n+1$ and $2n-1$. Otherwise, $p|(2n+1)-(2n-1) = 2$, which cannot be since $(2m+1)^2$ is odd.

We now have two possible cases (a and b are integers): $2n+1 = a^2$ and $2n-1 = 3b^2$, or $2n+1 = 3a^2$ and $2n-1 = b^2$. The former case implies $2n = a^2 - 1 = 3b^2 + 1$ or $a^2 = 3b^2 + 2 \equiv 2 (\bmod 3)$, which is impossible since for any integer a, $a^2 \equiv 0$ or 1 modulo 3.

Therefore, $2n - 1 = b^2$ for some odd positive integer b. Let $k = \dfrac{b-1}{2}$. Then

$$k^2 + (k+1)^2 = 2k^2 + 2k + 1 = \frac{(2k+1)^2 + 1}{2} = \frac{b^2+1}{2} = n.$$

The problem was also solved by:

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