PROBLEM OF THE WEEK

Solution of Problem No. 9 (Spring 2010 Series)

Problem: Let f be a given nonconstant polynomial with integer coefficients.

Show that for infinitely many primes p_i there is at least one corresponding integer x_i with p_i a factor of $f(x_i)$.

Solution (by Tom Engelsman, Chicago, IL)

CASE I ($a_0 = 0$):

If the constant coefficient a_0 is zero, then for infinitely many primes p_i, one just chooses $x_i = p_i$, which simply yields:

$$f(x_i) \equiv 0 (\operatorname{mod} p_i);$$

i.e., p_i is a factor of $f(x_i)$.

CASE II ($a_0 \neq 0$):

If the constant coefficient a_0 is nonzero, then let's consider for any integer t a transformed polynomial:

$$f(a_0 t\, x) = \sum_{k=0}^{n} a_k (a_0 t\, x)^k = a_0 \left[1 + \sum_{k=1}^{n} a_k a_0^{k-1} t^k x^k \right] = a_0 \cdot g(x)$$

There exists some prime p such that $g(\beta) \equiv 0 (\operatorname{mod} p)$ for some integer β, since g can take on the values $0, \pm 1$ at only a finite number of points. Since $g(\beta) \equiv 1 (\operatorname{mod} t)$, this shows that $\gcd\{p, t\} = 1$. Thus one now obtains:

$$f(a_0 t \beta) \equiv 0 (\operatorname{mod} p)$$

Since t is any arbitrary integer, infinitely many primes p must occur in this way.

Also completely or partially solved by:

Graduates: Tairan Yuwen (Chemistry)

Others: Siavash Ameli (Grad. student, Toosi Univ. of Tech, Iran), Gruian Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Sorin Rubinstein (TAU faculty, Israel), William Wu (Grad. student, Stanford Univ.), Zhengpeng Wu (Engr. Tsinghua Univ. China), Hyung-bin Youk (MD. Korea)