PROBLEM OF THE WEEK

Solution of Problem No. 1 (Spring 2011 Series)

Problem: Let $x_1, x_2, \ldots, x_{2011}$ be real numbers. For which real value(s) of c is

$$|x_1 - c| + |x_2 - c| + \cdots + |x_{2011} - c|$$

minimum?

Solution: (by Yixin Wang, Sophomore, ECE)

Let's look at the quantity $|x_m - c| + |x_n - c|$, where $x_m \leq x_n$.
If $c < x_m \leq x_n$, then $|x_m - c| + |x_n - c| = x_m + x_n - 2c > x_n - x_m$.
If $x_m \leq c \leq x_n$, then $|x_m - c| + |x_n - c| = x_n - x_m$.
If $x_m \leq x_n < c$, then $|x_m - c| + |x_n - c| = 2c - x_m - x_n > x_n - x_m$.

This tells us that $|x_m - c| + |x_n - c|$ reaches its minimum value when $x_m \leq c \leq x_n$. Back to the problem: WLOG, assume that $x_1 \leq x_2 \leq \cdots \leq x_{2011}$. Now, group the terms in the problem's equation as follows:

$$(|x_1 - c| + |x_{2011} - c|) + (|x_2 - c| + |x_{2010} - c|) + \cdots + (|x_{1005} - c| + |x_{1007} - c|) + |x_{1006} - c|$$

Note that if we set $c = x_{1006}$, each of the 1006 quantities in the above equation will reach their respective minimum values. Since the quantity $|x_{1006} - c|$ reaches its minimum at only that point, $c = x_{1006}$ is the only point the minimum of the whole equation is reached. So, the answer is c equals the median of the set $\{x_1, x_2, \ldots, x_{2011}\}$.

The problem was also solved by:

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