PROBLEM OF THE WEEK

Solution of Problem No. 10 (Spring 2011 Series)

**Problem:** Show that the boundary value problem $y'' + p(x)y' - \lambda^2 y = 0$, $y(a) = y(b) = 0$, $a \neq b$, $p(x)$ an arbitrary continuous function on $[a, b]$, does not have a nontrivial solution for any real $\lambda$.

**Solution:** (by Thierry Zell, Faculty at Lenoir–Rhyne University)

Suppose that the boundary value problem

$$y'' + p(x)y' - \lambda^2 y = 0, \quad y(a) = y(b) = 0; \tag{1}$$

does have a non-trivial solution $y \not\equiv 0$. Since $-y$ is also a non-trivial solution of (1), we can assume without loss of generality that our solution $y$ has a positive global maximum $M = y(c)$ on $[a, b]$.

We must have $y'(c) = 0$, so that

$$y''(c) = \lambda^2 M. \tag{2}$$

In the case where $\lambda \neq 0$, we already have a contradiction since $y''(c) > 0$ would violate the second derivative test.

But if $\lambda = 0$, our original problem (1) becomes a first–order linear differential equation for $y'$.

We must have:

$$y'(x) = y'(0) \exp\left( \int_0^x \Big( -p(t) \Big)\, dt \right). \tag{3}$$

This expression implies that $y'$ is either identically zero, or never vanishes; in either ease, this gives a contradiction.

**The problem was also solved by:**

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