PROBLEM OF THE WEEK

Solution of Problem No. 13 (Spring 2011 Series)

Problem: Suppose $P(x)$ is a real polynomial of degree $k \geq 1$. Show that the power series expansion for $f(x) = e^{P(x)}$ about any point x_0, cannot have k consecutive zero coefficients.

Solution: (by Sorin Rubinstein, TAU faculty, Israel)

Suppose that the power series expansion

$$y = e^{P(x)} = e^{P(x_0)} + P'(x_0)e^{P(x_0)}(x - x_0) + \cdots$$

of $y = e^{P(x)}$ about x_0 has k consecutive 0 coefficients. Then, since $e^{P(x_0)} \neq 0$, there exists some nonzero polynomial $Q(x)$ such that
$y = Q(x) + (x - x_0)^{n+k+1}S(x)$ where $n = \deg(Q(x))$ and $S(x) \in C^\infty(R)$.
We plug this form into the evident relation $y' = P'(x)y$ and obtain:

$$Q'(x)+(n+k+1)(x-x_0)^{n+k}S(x)+(x-x_0)^{n+k+1}S'(x) = P'(x)Q(x)+P'(x)(x-x_0)^{n+k+1}S(x)$$

Thus

$$P'(x)Q(x) - Q'(x) = (x - x_0)^{n+k}T(x) \tag{1}$$

where $T(x) = (n + k + 1)S(x) + (x - x_0)(S'(x) - P'(x)S(x)) \in C^\infty(R)$. But this is impossible since the left hand side of the identity (1), a non-zero polynomial of degree $n + k - 1$, cannot have a zero of multiplicity $n + k$.

The problem was also solved by:

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