PROBLEM OF THE WEEK

Solution of Problem No. 14 (Spring 2011 Series)

Problem: Let f be a function on [0,2] such that $f(x) > 0$ and $f''(x) \geq 0$ for all x and

$$\int_0^1 f(t)\,dt \cdot \int_1^2 \frac{dt}{f(t)} \leq 1.$$

Show that

$$\int_0^2 f(t)dt \leq 2f(2).$$

Solution: (by Bennett Marsh, Marmion Academy, Aurora, IL)

Since $f''(x)$ is defined for all x, $f(x)$ is continuous on $[0,2]$. Since it is a continuous function, the average value of $f(x)$ on $[0,1]$ is $f(a)$, where $0 \leq a \leq 1$. Similarly, the average value of $\dfrac{1}{f(x)}$ on $[1,2]$ is $\dfrac{1}{f(b)}$, where $1 \leq b \leq 2$. By the definition of average value,

$$\int_0^1 f(t)\,dt \cdot \int_1^2 \frac{dt}{f(t)} = \frac{f(a)}{f(b)} \leq 1.$$

Thus, $f(a) \leq f(b)$. Because of this, $f'(x)$ must be positive at some point in $[a,b]$. Since $f''(x) \geq 0$, the slope must remain positive from that point to $x = 2$, so $f(a) \leq f(b) \leq f(2)$. Again, since $f''(x) \geq 0$, $f(x)$ must be less than or equal to $f(b)$ for all x in $[a,b]$, so $f(b)$ must be the maximum of f on $[a,b]$ and thus also on $[1,b]$. Since $f(2) \geq f(b)$, $f(2)$ must be the maximum of f on $[1,2]$. From all of this we get

$$\int_0^1 f(t)\,dt = f(a) \leq f(2) \quad \text{and} \quad \int_1^2 f(t)\,dt \leq f(2).$$

Thus,

$$\int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt \leq 2f(2).$$

The problem was also solved by:

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