PROBLEM OF THE WEEK

Solution of Problem No. 4 (Spring 2011 Series)

Problem: Show that four consecutive binomial coefficients

$$\binom{n}{r}, \quad \binom{n}{r+1}, \quad \binom{n}{r+2}, \quad \binom{n}{r+3}$$

(with n, r positive and $r+3 \leq n$) can never be in arithmetic progression.

Solution: (by Steven Landy, IUPUI Physics staff)

If we try to find three terms in arithmetic progression we need,

$$\binom{n}{r} - \binom{n}{r+1} = \binom{n}{r+1} - \binom{n}{r+2}. \tag{1}$$

When (1) is expanded into factorials and reduced we find

$$(r+1)(r+2) - 2(n-r)(r+2) + (n-r)(n-r-1) = 0. \tag{2}$$

Equation (2) may be solved as a quadratic for either n or r giving

$$n = \frac{1}{2}(4r + 5 \pm \sqrt{8r+17}) \quad \text{and} \quad r = \frac{1}{2}(n - 2 \pm \sqrt{n+2}).$$

So there are many solutions for three consecutive binomial coefficients in arithmetic progression, for example $n = 7, r = 1$ or 4. These are actually the same three numbers but in reverse order. A "four in a row" would require a second "three in a row" beginning one space to the right of another three in a row. Since for any legal n there are only two acceptable r values (which are not consecutive) there can never be four consecutive binomial coefficients in arithmetic progression.

The problem was also solved by:

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