PROBLEM OF THE WEEK
Solution of Problem No. 7 (Spring 2011 Series)

**Problem:** Show that

$$\frac{1}{2\sqrt{n}} < \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} < \frac{1}{\sqrt{2n+1}}$$

for every $n = 2, 3, \ldots$.

**Solution:** (by Richard Eden, Math Graduate student, Purdue University)

For any $k > \frac{1}{2}$,

$$\frac{2k-1}{2k} < \frac{\sqrt{2k-1}}{\sqrt{2k+1}} \iff \sqrt{2k-1}\sqrt{2k+1} < 2k \iff 4k^2-1 < 4k^2,$$

and the last inequality is ture. As $k$ runs through the integers from 1 to $n$,

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n} < \frac{\sqrt{1}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{7}}\cdots\frac{\sqrt{2n-1}}{\sqrt{2n+1}} = \frac{1}{\sqrt{2n+1}}.$$

For any $k > 1$,

$$\frac{\sqrt{k-1}}{\sqrt{k}} < \frac{2k-1}{2k} \iff 2\sqrt{k}\sqrt{k-1} < 2k-1 \iff 4k^2-4k < 4k^2-4k+1,$$

where the last inequality is again true. As $k$ runs through the integers from 2 to $n$,

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n} > \frac{1}{2}\cdot\frac{\sqrt{1}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{3}}\cdot\frac{\sqrt{n-1}}{\sqrt{n}} = \frac{1}{2\sqrt{n}}.$$

**The problem was also solved by:**

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