PROBLEM OF THE WEEK

Solution of Problem No. 12 (Spring 2012 Series)

Problem: If two balls are chosen one at a time at random from an n dimensional ball B, the probability that the ball with center the first point and radius equal to the distance between the two points lies completely inside B equals $(n!)^2/(2n)!$. Derive this formula for the cases $n = 1, 2$, and 3. (A ball in one dimension is just a line segment, and the radius of a line segment is half its length. In two dimensions a ball is a disc.)

Solution: (by Bennett Marsh, Freshman, Engineering, Purdue University)

First consider $n = 1$. The probability formula gives a probability of $(1!)^2/(2)! = 1/2$. The ball is simply a line segment of length $2R$. If the first point is at a distance r from the center of the segment, then the second point must fall within a ball of radius $R - r$ centered at the first point. Thus, given the first point, the probability of the second point satisfying the condition is $(R-r)/R$. The probability density function of the first point as a function of r is just the constant function $p(r) = 1/R$, with $0 \leq r \leq R$. The probability of choosing two points that satisfy the condition is then just the integral of the product of the 2 probability functions, given by

$$\int_0^R p(r)\frac{R-r}{R}dr = \frac{1}{R^2}\int_0^R (R-r)dr = \frac{1}{R^2}\left(R^2 - \frac{R^2}{2}\right) = \frac{1}{2}.$$

For $n = 2$, the formula gives $(2!)^2/(4)! = 1/6$. We can proceed in much the same way, with R being the radius of the disk and r the distance of the first point from the center. The probability of choosing the second point within an acceptable region is given by $\pi(R-r)^2/\pi R^2$, and the probability density function of the first point changes to $p(r, \theta) = 1/(\pi R^2)$. We can integrate the product again, this time over the entire disk (in polar coordinates):

$$\int_0^{2\pi}\int_0^R p(r,\theta)\frac{(R-r)^2}{R^2}r\,dr\,d\theta = \frac{2}{R^4}\int_0^R (R^2r - 2Rr^2 + r^3)dr = 2\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{1}{6}.$$

Finally, for $n = 3$, the formula gives $(3!)/(6)! = 1/20$. The probability of choosing an acceptable 2nd point is $(R-r)^3/R^3$, and the PDF of the first point is $p(r, \theta, \phi) = 1(4/3\pi R^3)$. Integrating in spherical coordinates, we get

$$\int_0^{\pi}\int_0^{2\pi}\int_0^R p(r,\theta,\phi)\frac{(R-r)^3}{R^3}r^2\sin\phi\,dr\,d\theta\,d\phi = \frac{3}{R^6}\int_0^R r^2(R-r)^3\,dr = 3\left(\frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6}\right) = \frac{1}{20}.$$

The problem was also solved by:

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