PROBLEM OF THE WEEK
Solution of Problem No. 14 (Spring 2012 Series)

Problem: An urn has four balls numbered 1, 2, 3, 4. They are drawn one at a time at random with replacement, that is, a ball is drawn, its number is noted, and the ball is replaced and the urn is mixed before the next draw. The draws continue until a number is drawn that is smaller than a previously drawn number. Find the probability that the last number drawn is 1.

Solution 1: (by Steve Spindler, Chicago)

Let p_n denote the probability that the last number drawn will be 1 given that the first number drawn is n. Since each of the numbers 1 to 4 is equally likely to be drawn, the total probability p that the last number drawn is 1 is: $p = (1/4)(p_1 + p_2 + p_3 + p_4)$.

Choose $2 \le r \le 4$ and assume the first number drawn is r. If the second number drawn is $r \le s \le 4$, then drawing terminates whenever a number less that s is drawn. So the probability that the last number drawn will be 1 is p_s. If the second number drawn is $2 \le s \le r$, this represents a decrease, and drawing terminates with a last number other than 1, so the probability that the last number drawn will be 1 is 0. Finally, if $s = 1$, drawing terminates at 1 and the probability that the last number drawn will be 1 is 1.

Summarizing, we have the following relations: $p_r = (1/4)(1 + p_r + p_{r+1} + \cdots + p_4)$ for $2 \le r \le 4$. Lastly suppose that 1 is drawn first. This doesn't affect future drawings in any way; it is as if the first drawing were a "pass". So $p_4 = p = (1/4)(p_1 + p_2 + p_3 + p_4)$.

Recapping, we have 4 equations:

$$\begin{aligned}
p_4 &= (1/4)(p_4 + 1) \\
p_3 &= (1/4)(p_4 + p_3 + 1) \\
p_2 &= (1/4)(p_4 + p_3 + p_2 + 1) \\
p_1 &= (1/4)(p_4 + p_3 + p_2 + p_1)
\end{aligned}$$

These are easily solved with the following results:
$p_4 = 1/3, \quad p_2 = 4/9, \quad p_3 = 16/27, \quad$ and $p_1 = p = 37/81$.

Solution 2: (by Sorin Rubinstein, Tel Aviv, Israel)

Let us denote by $\underline{\mathbf{a}}$ any maximal sequence of consecutive appearances of the ball numbered $\mathbf{a}$. By maximality we mean that the ball drawn before this sequence – if any, and the ball drawn after this sequence – if any, are different from $\mathbf{a}$. The probability that the draw starts with an infinite sequence $\underline{\mathbf{a}}$ is 0, and the probability that the draw starts with a finite maximal sequence $\underline{\mathbf{a}}$ is independent of the value of $\mathbf{a}$ and therefore equal to $\frac{1}{4}$. The probability that a finite sequence $\underline{\mathbf{a}}$ is followed by an infinite sequence $\underline{\mathbf{b}}$ equals 0, and the probability that a finite maximal sequence $\underline{\mathbf{a}}$ is followed by a finite maximal sequence $\underline{\mathbf{b}}$ is independent of the value of $\mathbf{b}$ and equal to $\frac{1}{3}$ since, by the maximality of $\underline{\mathbf{a}}, \mathbf{b}$ must be different from $\mathbf{a}$. The sequences of numbers that satisfy the condition of theproblem are (not necessarily strictly) increasing sequences different from $\underline{\mathbf{1}}$ followed by an $\mathbf{1}$.

The possibilities are $\underline{\mathbf{2}}, \mathbf{1}$; $\underline{\mathbf{3}}, \mathbf{1}$; $\underline{\mathbf{4}}, \mathbf{1}$; with a probability of $\frac{1}{4} \cdot \frac{1}{3}$ each, $\underline{\mathbf{1}}, \underline{\mathbf{2}}, \mathbf{1}$; $\underline{\mathbf{1}}, \underline{\mathbf{3}}, \mathbf{1}$; $\underline{\mathbf{1}}, \underline{\mathbf{4}}, \mathbf{1}$; $\underline{\mathbf{2}}, \underline{\mathbf{3}}, \mathbf{1}$; $\underline{\mathbf{2}}, \underline{\mathbf{4}}, \mathbf{1}$; $\underline{\mathbf{3}}, \underline{\mathbf{4}}, \mathbf{1}$ with a probability of $\frac{1}{4} \cdot \left(\frac{1}{3}\right)^2$ each, $\underline{\mathbf{1}}, \underline{\mathbf{2}}, \underline{\mathbf{3}}, \mathbf{1}$; $\underline{\mathbf{1}}, \underline{\mathbf{2}}, \underline{\mathbf{4}}, \mathbf{1}$; $\underline{\mathbf{1}}, \underline{\mathbf{3}}, \underline{\mathbf{4}}, \mathbf{1}$; $\underline{\mathbf{2}}, \underline{\mathbf{3}}, \underline{\mathbf{4}}, \mathbf{1}$ with a probability of $\frac{1}{4} \cdot \left(\frac{1}{3}\right)^3$ each, and $\underline{\mathbf{1}}, \underline{\mathbf{2}}, \underline{\mathbf{3}}, \underline{\mathbf{4}}, \mathbf{1}$ with a probability of $\frac{1}{4} \cdot \left(\frac{1}{3}\right)^4$. Thus the total probability is

$$3 \cdot \frac{1}{4} \cdot \frac{1}{3} + 6 \cdot \frac{1}{4} \cdot \left(\frac{1}{3}\right)^2 + 4 \cdot \frac{1}{4} \cdot \left(\frac{1}{3}\right)^3 + \frac{1}{4} \cdot \left(\frac{1}{3}\right)^4 = \frac{37}{81}$$

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