PROBLEM OF THE WEEK

Solution of Problem No. 2 (Spring 2012 Series)

Problem: Find $\lim\limits_{n\to\infty} \dfrac{1+2^2+3^3+\cdots+(n-1)^{n-1}+n^n}{n^n}$.

Solution: (by Steve Spindler, Chicago)

Let $S_n = \dfrac{\sum_{k=1}^n k^k}{n^n}$. Clearly, $S_n \geq 1$. And $1 \leq k \leq n \implies k^k \leq n^k$, so

$$\begin{aligned} S_n \leq \frac{\sum_{k=1}^n n^k}{n^n} &= \sum_{j=0}^{n-1} \left(\frac{1}{n}\right)^j \\ &= \frac{(1/n)^n - 1}{(1/n) - 1} \\ &= R_n, \end{aligned}$$

a simple geometric series. Obviously $\lim\limits_{n\to\infty} R_n = 1$. Since $1 \leq S_n \leq R_n$, it follows that $\lim\limits_{n\to\infty} S_n = 1$.

Remark: It is true that the limit of a sum of ten numbers is the sum of the limits, but the analog for infinitely many numbers need not hold. A number of proposed solutions failed for this reason.

The problem was also solved by:

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Others: Lycee Jaques Audiberti (Antibes, France), Manuel Barbero (New York), Hongwei Chen (Faculty, Christopher Newport U. VA), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Elie Ghosn (Montreal, Quebec), John Karpis (Miami Springs, FL), Rob Klein (West Lafayette, IN), Anastasois Kotronis (Athens, Greece), Chris Kyriazis (High school teacher, Chalki, Greece), Steven Landy (Physics Faculty, IUPUI), Angel Plaza (ULPGC, Spain), Jason Rahman (High School Senior, Hazleton, IN), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Postdoc. UCLA), Pawan Singh (United

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