PROBLEM OF THE WEEK
Solution of Problem No. 6 (Spring 2012 Series)

Problem: A,B,C,D are four distinct points in three space. Suppose each of the angles A$\underline{\text{B}}$C, B$\underline{\text{C}}$D, C$\underline{\text{D}}$A, and D$\underline{\text{A}}$B are right angles. Show that all four points lie in the same plane.

Solution 1: (by Kilian Cooley, Junior, Math & AAE, Purdue University)

Assume without loss of generality that points A, B, and C are located along the y-axis, at the origin, and along the x-axis respectively of some coordinate system. Denote by $\vec{r}^{PQ}$ the position vector from point P to point Q, which are expressed as column vectors. Thus, using the fact that angles DAB and BCD are right angles:

$$\begin{aligned}\vec{r}^{BD} = \vec{r}^{BA} + \vec{r}^{AD} &= \vec{r}^{BC} + \vec{r}^{CD} \\ \vec{r}^{BA} \cdot \vec{r}^{AD} &= 0 \\ \vec{r}^{BC} \cdot \vec{r}^{CD} &= 0\end{aligned}$$

Since point A is located on the y-axis and C on the x-axis, the second and third relations can be written respectively as

$$\begin{bmatrix} 0 \\ y_A \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x_D - x_A \\ y_D - y_A \\ z_D - z_A \end{bmatrix} = 0$$
$$\begin{bmatrix} x_C \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x_D - x_C \\ y_D - y_C \\ z_D - z_C \end{bmatrix} = 0$$

From which we obtain

$$\begin{aligned} y_D - y_A &= 0 \\ x_D - x_C &= 0 \end{aligned}$$

So point D lies along a line parallel to the z-axis through $\begin{bmatrix} x_C \\ y_A \\ 0 \end{bmatrix}$. We want to show now that $z_D = 0$. Since CDA is a right angle, it follows that

$$r^{AD} \cdot r^{CD} = 0$$

$$\begin{bmatrix} x_D - x_A \\ y_D - y_A \\ z_D - z_A \end{bmatrix} \cdot \begin{bmatrix} x_D - x_C \\ y_D - y_C \\ z_D - z_C \end{bmatrix} = \begin{bmatrix} x_D - x_A \\ 0 \\ z_D - z_A \end{bmatrix} \cdot \begin{bmatrix} 0 \\ y_D - y_C \\ z_D - z_C \end{bmatrix} = (z_D - z_A)(z_D - z_C) = 0$$

Since points A and C lie in the same plane by definition, $z_A = z_C = 0$. Hence $z_D = 0$, and all four points lie in the same plane.

Solution 2: (by Steven Landy, Physics Faculty, IUPUI)

From perpendicularity we have, using vectors,

$$(B - A) \cdot (C - B) = 0 \quad \text{and} \quad (C - B) \cdot (D - C) = 0, \quad \text{so that}$$

$(C - B)$ is a multiple of $(B - A) \times (D - C)$ unless $(B - A) \times (D - C) = \vec{0}$. In the same way we see that
$(A - D)$ is a multiple of $(B - A) \times (D - C)$ unless $(B - A) \times (D - C) = \vec{0}$.

Therefore either $(C - B)$ is parallel to $(A - D)$, so these vectors are coplanar or $(B - A) \times (D - C) = \vec{0}$ so $(B - A)$ is parallel to $(D - C)$, which proves the theorem.

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