PROBLEM OF THE WEEK

Solution of Problem No. 7 (Spring 2012 Series)

Problem: Suppose that for every (including the empty set and the whole set) subset X of a finite set S there is a subset $\mathbf{X}^*$ of S and suppose that if X is a subset of Y then $\mathbf{X}^*$ is a subset of $\mathbf{Y}^*$. Show that there is a subset A of S satisfying $\mathbf{A}^* = \mathbf{A}$.

Solution 1: (by Sorin Rubinstein, Tel Aviv, Israel)

We define a sequence of subsets of S by: $S_0 = \phi$ and for every non-negative integer n, $S_{n+1} = S_n^*$. Since, clearly, $S_0 \subseteq S_1$ and $S_k \subseteq S_{k+1} \Rightarrow S_k^* \subseteq S_{k+1}^* \Rightarrow S_{k+1} \subseteq S_{k+2}$ this is an increasing sequence: $S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq \cdots$ of subsets of the finite set S. Hence there must exist an index n such that $S_n = S_{n+1}$. We define $A = S_n$. This ensures that $A^* = S_n^* = S_{n+1} = S_n = A$.

Solution 2: (by Sorin Rubinstein, Tel Aviv, Israel)

The condition that S is finite is not necessary and will not be used in this solution.

Let us define the set $\Omega = \{V \subseteq S : V \subseteq V^*\}$. Clearly $\phi \in \Omega$. We also define the set $A = \bigcup_{V \in \Omega} V$. If $V \in \Omega$, then $V \subseteq A$, which leads to $V^* \subseteq A^*$ and since also $V \subseteq V^*$, to $V \subseteq A^*$. Since this is true for every $V \in \Omega$, it follows that $A = \bigcup_{V \in \Omega} V \subseteq A^*$. Moreover, from $A \subseteq A^*$ it follows that $A^* \subseteq (A^*)^*$. Then, $A^* \in \Omega$ and, consequently, $A^* \subseteq \bigcup_{V \in \Omega} V = A$. Finally, from $A \subseteq A^*$ and $A^* \subseteq A$ follows that $A = A^*$.

The problem was also solved by:

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