## PROBLEM OF THE WEEK

Solution of Problem No. 10 (Spring 2013 Series)

Problem:
Start with some pennies. Flip each penny until a head comes up on that penny. The winner(s) are the penny(s) which were flipped the most times. Prove that the probability there is only one winner is at least $2 / 3$.

Solution: (by Steven Landy, Physics Faculty, IUPUI)
We may define $p(0)=0 . p(1)=1$. To calculate $p(2)$ we expand into cases. After one flip of two coins there must result either 0 , 1 , or 2 still "alive" (not yet heads). So expanding

$$
p(2)=1 / 4\left(\binom{2}{0} p(0)+\binom{2}{1} p(1)+\binom{2}{2} p(2)\right)
$$

which gives $p(2)=2 / 3$. Similarly we find $p(3)=5 / 7$ etc.
We prove by induction that for $n>2$ that $p(n) \geq 2 / 3$. Suppose that $p(k)$ is $\geq 2 / 3$ for $k=1,2, \ldots, n-1$. Then by expansion

$$
p(n)=\frac{1}{2^{n}}\left(\binom{n}{0} p(0)+\binom{n}{1} p(1)+\sum_{k=2}^{n-1}\binom{n}{k} p(k)+\binom{n}{n} p(n)\right)
$$

which is the same as

$$
\begin{aligned}
& \left(2^{n}-1\right) p(n)=1 p(0)+n p(1)+\sum_{k=2}^{n-1}\binom{n}{k} p(k) \geq n+\frac{2}{3} \sum_{k=2}^{n-1}\binom{n}{k}=n+\frac{2}{3}\left(2^{n}-1-n-1\right) \\
& \left(2^{n}-1\right) p(n) \geq \frac{2}{3}\left(2^{n}-1\right)+\frac{1}{3}(n-2) \text { and so } p(n) \geq 2 / 3
\end{aligned}
$$

## Remarks:

This problem is related to homeopathic medicine. See Homeopathic Dilution in Wikipedia. For example, if you start with a gallon of salt water, pour out half the well mixed solution and replace it with pure water, and repeat forever, with probability at least $2 / 3$ there is at some time exactly one salt molecule in the gallon.

The following, which is a rearrangement of some of the ideas in the correct solutions submitted, while perhaps not a proof, provides some intuition.

Toss the coins together, remove those which come up heads, and stop if either only one coin remains or no coins remain. Repeat until you stop. Since this procedure is certain to stop, the problem statement is equivalent to asking for a proof that the probability that exactly one coin remains when this process stops is at least twice the probability that no coin remains when it stops. But each time during this process that a collection of coins (say there are k coins) is flipped, the probability the process stops with that collective flip and exactly one coin remains (which is $k 2^{-k}$ ), is at least twice the probability that the process stops with that collective flip and no coins remain(which is $2^{-k}$ ).

## The problem was also solved by:

## Graduates: Tairan Yuwen (Chemistry)

Others: Siavash Ameli (Graduate Student, IIT, Chicago), Peter Kornya (Retired Faculty, Ivy Tech), Karthikeyan Marimuthu (Grad Student, Carnegie Mellon Univ.), Christopher Nelson (PostDoc, UCSD), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), David Stigant, Chris Willy (Adjunct faculty, George Washington Univ.)

