PROBLEM OF THE WEEK

Solution of Problem No. 11 (Spring 2013 Series)

Problem:

Let $c_0 > 0$, $c_1 > 0$, and $c_{n+1} = \sqrt{c_n} + \sqrt{c_{n-1}}$, $n \geq 1$.
Show that $\lim\limits_{n\to\infty} c_n$ exists and find this limit.

Solution: (by Julien Bureaux, Paris, France)

Let $c_0 > 0$, $c_1 > 0$, and

$$c_{n+1} = \sqrt{c_n} + \sqrt{c_{n-1}}, \quad n \geq 1 \tag{1}$$

Show that $\lim\limits_{n\to\infty} c_n$ exists and find this limit.
We will prove that

$$\limsup c_n \leq 4 \leq \liminf c_n \tag{2}$$

First remark that the sequence $b_n = \max\{4, c_n, c_{n-1}\}$ is non-increasing. Indeed, the trivial lower bound $b_n \geq 4$ yields $c_{n+1} \leq 2\sqrt{b_n} \leq b_n$; we conclude with $b_{n+1} = \max\{4, c_{n+1}, c_n\} \leq \max\{4, b_n, b_n\} = b_n$. As a consequence, an upper bound for c_n is $\max\{4, c_0, c_1\}$. In the same way, $c_n \geq \min\{4, c_0, c_1\}$.

These bounds show that both $\liminf c_n$ and $\limsup c_n$ lie in $(0, \infty)$. Furthermore we deduce from (1) that

$$\liminf c_n \geq 2\sqrt{\liminf c_n}, \qquad \limsup c_n \leq 2\sqrt{\limsup c_n}$$

This proves (2), hence the result.

The problem was also solved by:

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